Some Rigidity and Dynamical Stability Results of Calibrated Submanifolds

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Overview

1. **Special Holonomy and Calibrated Submanifold**
2. **Minimal Submanifold and Mean Curvature Flow**
3. **Bundle Manifolds with Special Holonomy**
4. **Main Results**

This is based on joint works with Mu-Tao Wang (Columbia University) and Ping-Hung Lee (Columbia University).

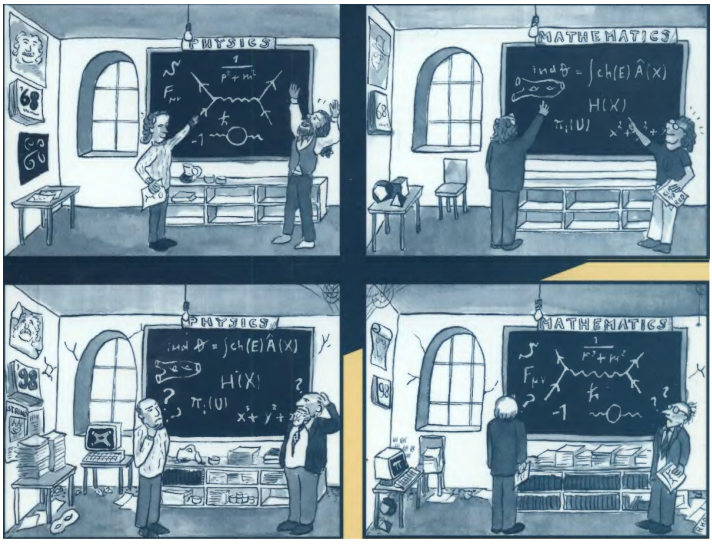


Figure: a drawing of Dijkgraaf. 1968 $\rightarrow$ 1998 $\rightarrow$ 2028?

Table of Contents

1. Special Holonomy and Calibrated Submanifold

2. Minimal Submanifold and Mean Curvature Flow

3. Bundle Manifolds with Special Holonomy

4. Main Results

Best Representative

General Question

For a Riemannian manifold $(M, g = g_{ij}\mathrm{d}x^i\mathrm{d}x^j)$ and a homology class $\xi \in \mathrm{H}_k(M)$, what is the best representative of ξ? Can it be represented by minimal submanifold?

The dual question is easier to handle: by the Hodge theorem, any cohomology class admits a unique harmonic representative.

This general question is rather difficult. One of the main interests is to investigate this question on manifolds with special holonomy. The purpose is twofold.

Special Holonomy

In 1955, Berger classified Riemannian holonomy groups. We will focus on the followings.

Dimension	Model Space	Holonomy	Parallel Forms
$2n$	$\mathbb{R}^{2n} \cong \mathbb{C}^n$	$\mathrm{SU}(n)$ (Calabi-Yau)	ω and Ω
$4n$	$\mathbb{R}^{4n} \cong \mathbb{H}^n$	$\mathrm{Sp}(n)$ (hyper-Kähler)	ω_I, ω_J and ω_K
7	$\mathbb{R}^7 \cong \mathrm{Im}\,\mathbb{O}$	G_2	φ and $\psi = *\varphi$
8	$\mathbb{R}^8 \cong \mathbb{O}$	Spin(7)	Φ

It turns out that they are all Ricci flat.

Except the Calabi-Yau's, there is no general existence criterion for other geometries.

Linear Models of Parallel Forms

For Calabi-Yau, the models are

$$\omega = \mathrm{d}x^1 \wedge \mathrm{d}y^1 + \cdots + \mathrm{d}x^n \wedge \mathrm{d}y^n$$

and

$$\Omega = (\mathrm{d}x^1 + i\mathrm{d}y^1) \wedge \cdots \wedge (\mathrm{d}x^n + i\mathrm{d}y^n) \ .$$

For G_2, the model is

$$\begin{aligned}\varphi = \mathrm{d}x^{456} &- \mathrm{d}x^4 \wedge (\mathrm{d}x^{01} + \mathrm{d}x^{23}) \\ &- \mathrm{d}x^5 \wedge (\mathrm{d}x^{02} + \mathrm{d}x^{31}) - \mathrm{d}x^6 \wedge (\mathrm{d}x^{03} + \mathrm{d}x^{12}) \ .\end{aligned}$$

Calibrated Submanifolds

Since $\mathrm{d}\cdot = \mathrm{alt}(\nabla\cdot)$, parallel forms are closed. The above parallel forms are more or less equivalent to the metric, and has comass one. In particular, they are calibrations defined by Harvey and Lawson.

A k-form Θ is called a calibration if $\mathrm{d}\Theta = 0$ and $|\Theta(L)| \leq 1$ for any k-plane L in the tangent space of any point.

Calibrated Submanifolds (cont'd)

A k-dimensional submanifold Σ is said to be calibrated by Θ

$$\Theta(T\Sigma) \equiv 1 \quad \Leftrightarrow \quad \Theta|_\Sigma = \mu_\Sigma \ \text{(the volume form of } \Sigma) \ .$$

It follows that Σ minimizes the volume within its homology class: for $\partial Y = \Sigma - \tilde{\Sigma}$

$$\text{vol}(\Sigma) = \int_\Sigma \Theta = \int_Y d\Theta + \int_{\tilde{\Sigma}} \Theta \leq \text{vol}(\tilde{\Sigma}) \ .$$

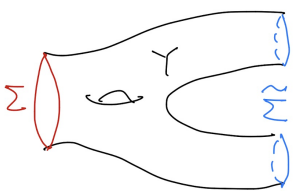


Figure: $\partial Y = \Sigma - \tilde{\Sigma}$

Special Holonomy and Calibration

Main interest for Differential Geometers:

Submanifold	**Dimension**	**Ambient**	**Calibration**
special Lagrangian	n	Calabi-Yau	$\mathrm{Re}(e^{-i\theta}\Omega)$
coassociative	4	G_2	φ
associative	3	G_2	$\psi = *\varphi$
Cayley	4	Spin(7)	Φ

Rigidity, Deformation and Moduli

The calibration argument implies the following: if Σ and $\tilde{\Sigma}$ are homologous and calibrated, then they have the same volume.

Rigidity Question

- When is a calibrated submanfiold unique?
- The deformation theory of calibrated submanifolds is known. What about the geometric property of those submanifolds? For instance, intersection between Σ and $\tilde{\Sigma}$?

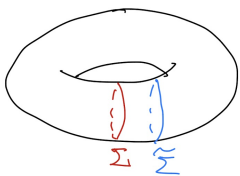


Figure: non-unique representatives

Table of Contents

1. Special Holonomy and Calibrated Submanifold

2. Minimal Submanifold and Mean Curvature Flow

3. Bundle Manifolds with Special Holonomy

4. Main Results

Mean Curvature Vector

For a submanifold $\Sigma \subset (M, g)$, the first variational formula of the volume reads

$$\delta_V \mathrm{vol}(\Sigma) = -\int_\Sigma \langle V, H_\Sigma \rangle \, \mathrm{d}\mu_\Sigma$$

where H_Σ is the mean curvature vector of Σ. Roughly speaking, H_Σ is the negative gradient of the volume functional.

Σ is minimal iff $H_\Sigma \equiv 0$. In other words, Σ is the critical state of the volume functional.

Calibrated submanifolds are not only critical states, but volume minimizers.

Mean Curvature Flow

On the other hand, we can consider the negative gradient flow of the volume functional starting from some initial Σ_0:

$$\frac{\partial \Sigma_t}{\partial t} = H_{\Sigma_t} \quad \text{with given initial } \Sigma_0 \ .$$

We hope that the flow exists for all time, and Σ_t converges to a minimal submanifold as $t \to \infty$.

However, this is not always the case, and the analysis is rather difficult. There are more results in the hypersurface case, and much less is known in higher codimensions.

Examples of Singularities

The mean curvature flow of $S^n(r)$ in $\mathbb{R}^{n+1}$ is given by:

$$r(t) = \sqrt{r_0^2 - 2nt}\ .$$

It collapses to a point at time $r_0^2/2n$.

There could be more complicated singularity formation during the mean curvature flow.

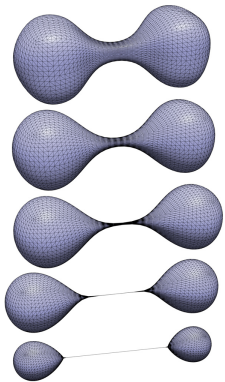


Figure: Neckpinch

More Illustrations of Singularities

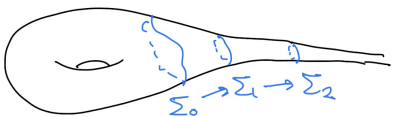


Figure: no limit as $t \to \infty$

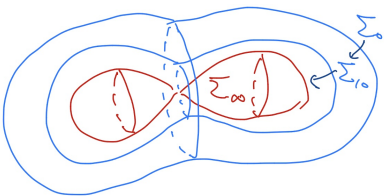


Figure: singularity at infinite time

Equation in Graphical Setting

Given $(f_1, \ldots, f_m) : \mathbb{R}^n \times [0, T) \to \mathbb{R}^m$, its graph is a one-parameter family of n-dimensional submanifold is $\mathbb{R}^{n+m}$: $(x^1, \ldots, x^n, f_1(x, t), \ldots, f_m(x, t))$.

In this setting, the mean curvature flow equation becomes

$$\frac{\partial f_\mu}{\partial t} = g^{ij} \frac{\partial^2 f_\mu}{\partial x^i \partial x^j}$$

for $\mu = 1, \ldots, m$, where g^{ij} is the inverse of $\delta_{ij} + \sum_\nu \partial_i f_\nu \partial_j f_\nu$.

Note that g^{ij} involves all components of f.

Criterion of Singularity Formation

The pioneering work of mean curvature flow was established by Huisken in the 90's. In particular, $T < \infty$ is the maximal existence time if and only if

$$\lim_{t \to T^-} \sup_{\Sigma_t} |\mathrm{II}|^2 = \infty$$

where $\mathrm{II} = h_{ij}^{\mu} \, \mathrm{d}x^i \otimes \mathrm{d}x^j \otimes n_\mu$ is the second fundamental form of Σ_t.

Remark: The mean curvature vector is the trace of the second fundamental form. In the graphical setting, the second fundamental form is basically the Hessian of f_μ, for $\mu = 1, \ldots, m$.

Limit and Flow through Singularity

Stability Question and More

- Given a minimal Σ, will the mean curvature flow bring nearby submanifolds to it? If Σ is calibrated, it is more likely to be the case.
- More generally, even if $T = \infty$, when can we say something about $\lim_{t\to\infty} \Sigma_t$?
- Even if $T < \infty$, can we analyze the behavior of the flow at T, and construct a continuation of the flow after time T?

Table of Contents

1. Special Holonomy and Calibrated Submanifold

2. Minimal Submanifold and Mean Curvature Flow

3. Bundle Manifolds with Special Holonomy

4. Main Results

Important Examples

Here are some explicitly constructed metrics of special holonomy.

Name	Geometry	Space
Stenzel	Calabi-Yau	T^*S^n
Calabi	hyper-Kähler	$T^*\mathbb{CP}^n$
Bryant-Salamon	G_2	$\mathbb{S}(S^3)$, $\Lambda^2_-(S^4)$ and $\Lambda^2_-(\mathbb{CP}^2)$
Bryant-Salamon	Spin(7)	$\mathbb{S}_-(S^4)$
Atiyah-Hitchin	hyper-Kähler	degree -4 bundle over $\mathbb{CP}^1$

Important Examples (cont'd)

- They are all total space of a vector bundle over sphere or complex projective space.
- Zero sections are calibrated.
- They appear in the degeneration of metrics, and are useful in the gluing construction for more examples of manifolds with special holonomy.
- Cvetič, Gibbons, Lü and Pope studied L^2-harmonic forms on some of them, and used the L^2-harmonic forms to obtain the brane resolution and AdS/CFT correspondence.

Sketch of the Construction

Input:

1. a vector bundle $\mathbb{R}^m \to E \xrightarrow{\pi} B$;
2. a Riemannian metric g_B on B, and a bundle metric g_E on E;
3. a metric connection on E.

The procedure:

- Given an orthonormal trivialization $\{e_\mu\}$ for E, write out the connection 1-form:
$$\nabla e_\mu = A_\mu^\nu \, e_\nu \ .$$
- The orthonormal trivialization also induces coordinate for the fibers: $\{y^\mu\}$. Note that $g_E = \sum_\mu |\mathrm{d}y^\mu|^2$ over each $b_0 \in B$.
$(E_{b_0} = \pi^{-1}(b_0) \cong \mathbb{R}^m)$

Sketch of the Construction (cont'd)

- The connection is equivalent to a decomposition of the tangent space of E. This allows us to construct a metric on the total space of E:

$$\pi^*(g_B) + \sum_\mu (\mathrm{d}y^\mu + A^\mu_\nu y^\nu)^2 \ .$$

- Let $r = \sqrt{\sum_\mu (y^\mu)^2}$. Consider

$$\alpha^2(r)\, \pi^*(g_B) + \beta^2(r) \sum_\mu (\mathrm{d}y^\mu + A^\mu_\nu y^\nu)^2 \ .$$

  It turns out that the special holonomy equation becomes an ODE system on $\alpha(r)$ and $\beta(r)$.

The Picture for the Decomposition

The connection is more or less equivalent to an assignment of the lifting of tangent vectors on B to tangent vectors on E.

Table of Contents

1. Special Holonomy and Calibrated Submanifold

2. Minimal Submanifold and Mean Curvature Flow

3. Bundle Manifolds with Special Holonomy

4. Main Results

Rigidity Result

Theorem (A)

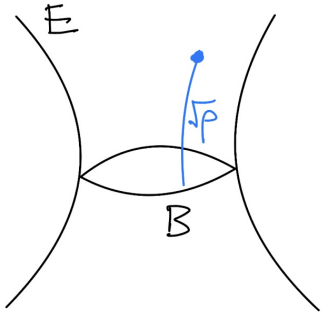
In the examples of previous section, the zero section is the only compact, minimal submanifold of dimension no less than its dimension. In other words, they are globally rigid.

Remark: For the Atiyah-Hitchin space, Bates and Montgomery proved that there exists closed geodesics which is not contained in the zero section.

Key Ingredient

Let ρ be the distance square to the zero section with respect to the special holonomy metric:

$$\rho = \left(\int \beta(r)\mathrm{d}r\right)^2 .$$



Key Ingredient (cont'd)

Let k be the dimension of the zero section B. The Hessian of ρ is k-convex: the sum of smallest k-eigenvalues of $\operatorname{Hess}\rho$ is non-negative. Moreover, it is strictly k-convex outside the zero section, $r > 0$.

If $\tilde{\tilde{\Sigma}}$ is a minimal submanifold of dimension no less than k, it follows that

$$\Delta_{\tilde{\tilde{\Sigma}}}\rho = \operatorname{tr}_{T\tilde{\tilde{\Sigma}}} \operatorname{Hess}\rho \geq 0 \ .$$

By the maximum principle, ρ must be a constant on $\tilde{\tilde{\Sigma}}$. The strictness implies that the constant can only be zero.

Dynamical Stability Result

Theorem (B)

In the examples of previous section, if one starts the mean curvature flow with a submanifold C^1-close to the zero section, the flow exists for all times. Moreover, it converges exponentially to the zero section as time goes to ∞.

Remark: The C^1 means the slope with respect to the zero section. It controls the induced metric, but not the second fundamental form. In the view point of the criterion of Huisken, this is a good condition. Requiring only C^0 is usually insufficient.

C^1 and C^0

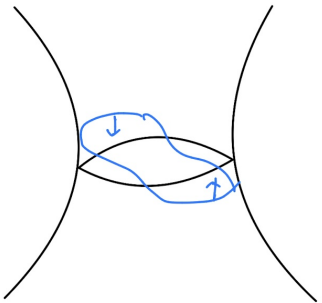


Figure: Theorem (B)

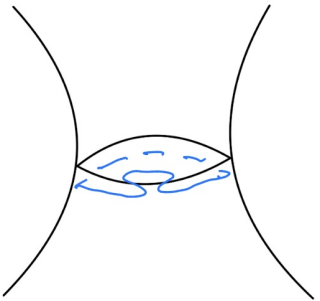


Figure: C^0-closeness is too weak

Caveat: The pictures are dimension-suppressed. Many bad behaviors will not happen for curves in surfaces.

Generalization

Later on, we are able to generalize both Theorem A and B to general settings.

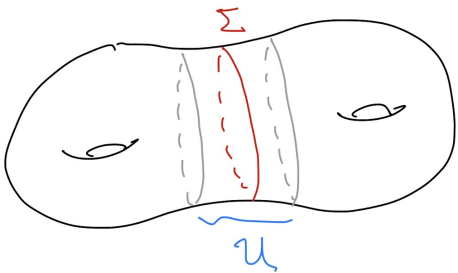
Theorem (C)

For a minimal submanifold Σ in a Riemannian manifold M, if the minimal submanifold is strongly stable, then there exists a tubular neighborhood $\mathcal{U}$ of Σ such that Theorem A and B hold true.

Strongly stable is a tensorial condition. It is slightly stronger than linear stability, $\delta_V^2 \mathrm{vol}(\Sigma) > 0$ for any nonzero variation V.

Illustration for Theorem (C)

We are able to give a precise estimate on the size of $\mathcal{U}$.



Special Lagrangian in Calabi-Yau

Let us focus on Calabi-Yau, $(M^{2n}, g, J, \omega, \Omega)$. A half-dimensional submanifold L is said to be special Lagrangian if it is calibrated by $\operatorname{Re}\Omega$, if and only if $\omega_L = 0$ (Lagrangian) and $\operatorname{Im}\Omega|_L = 0$ (special).

In this setting, the strongly stability condition on L is equivalent to that L has positive Ricci curvature.

In other words, a positively Ricci curved special Lagrangian is locally geometrically rigid, and dynamically stable.

More on the Lagrangians

From a symplectic perspective, a tubular neighborhood of a Lagrangian is the same as L in T^*L with $\omega = \sum_i \mathrm{d}q^i \wedge \mathrm{d}p_i$. In this way, all the geometry information is encoded in Ω.

The nearby Lagrangian submanifolds can be described by closed 1-forms. By Hodge theorem,

$$\{\text{closed 1-forms}\} = \{\text{harmonic 1-forms}\} \oplus \{\mathrm{d}u\} \ .$$

The first component is finite dimensional. The second component is infinite dimensional, and is referred as Hamiltonian deformations.

Dynamical Stability of Special Lagrangians

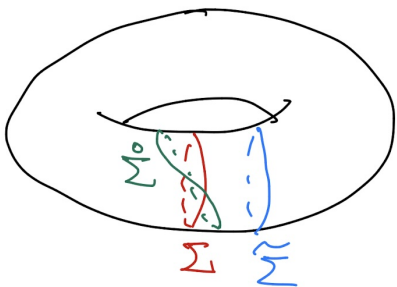
Theorem (D)

For a special Lagrangian L in a Calabi-Yau, the dynamical stability theorem holds true for C^1-close, Hamiltonian equivalent Lagrangian submanifold.

Remark: If we only considers Hamiltonian equivalent perturbations, the strongly stability condition is not needed.

Illustration for Theorem (D)

$\mathring{\Sigma}$ is hamiltonian equivalent to Σ, but $\tilde{\Sigma}$ is not.



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Thank you for your attention.