A diagrammatic view of Khovanov homology.

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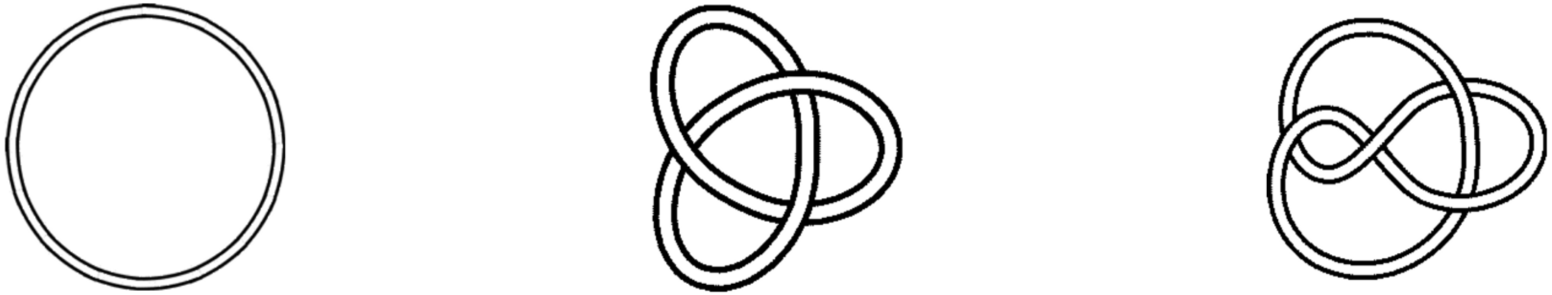
2025-08-21 @ The 2nd iTHEMS-NCTS Joint Workshop.

History

Mathematics		Physics
① Jones polynomial V.F. Jones, 1984	physical interpretation	② SU(2) Chern–Simons theory E. Witten, 1989
③ Quantum-invariants Reshetikhin–Turaev, 1990	mathematical formulation	
categorify ④ Khovanov homology M. Khovanov, 2000	physical interpretation	⑤ 4d, 5d Gauge theory E. Witten, 2011
⑦ Fukaya categories LePage – Shende, 2025	mathematical formulation	⑥ Homological Mirror symmetry Aganagic, 2022.

Contents.

Knots & knot invariants.



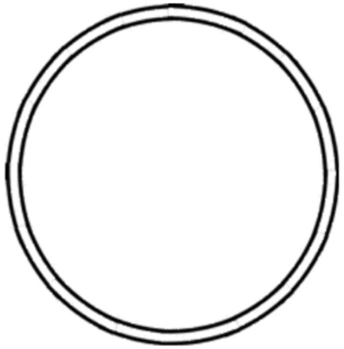
Q. How to distinguish knots?

Q. How to tell whether a knot can be unknotted?

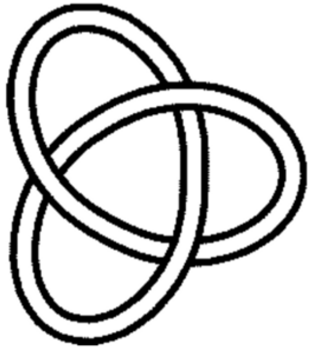
$\rightsquigarrow$ Find a good knot invariant φ, s.t.

$$\boxed{K \approx K' \implies \varphi(K) = \varphi(K')}$$

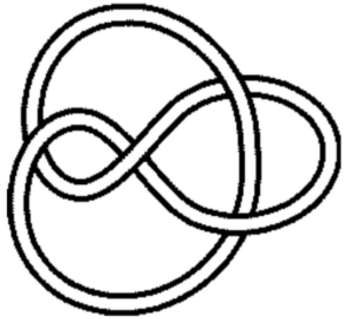
Jones polynomial (Jones, 1984)



$q + q^{-1}$



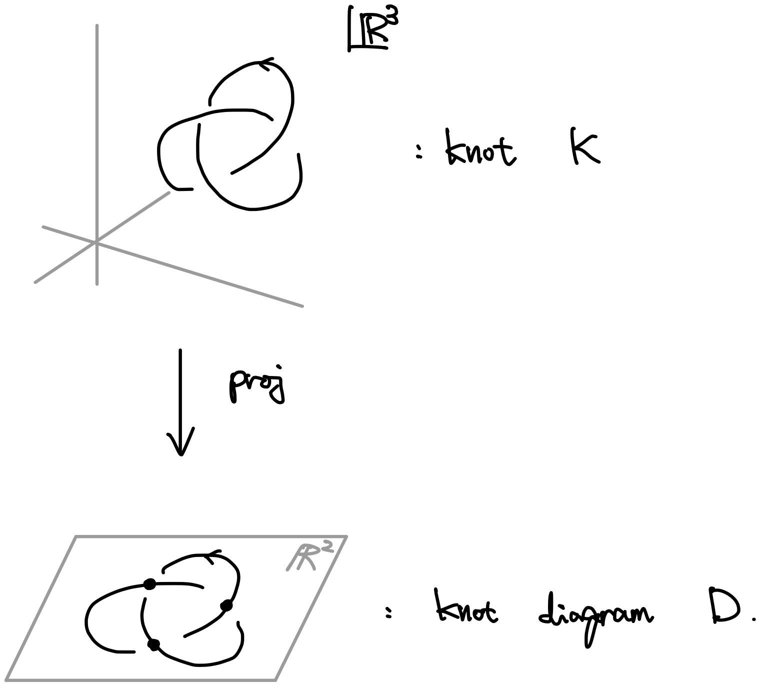
$-q^{-9} + q^{-5} + q^{-3} + q^{-1}$



$q^{-5} + q^{5}$

Given a knot diagram D,
it's <u>Jones polynomial</u> can be computed by the formula:

$$\begin{cases} 0) & \langle \phi \rangle = 1. \\ 1) & \langle \bigcirc\, D \rangle = (q + q^{-1}) \langle D \rangle \\ 2) & \langle \times \rangle = \langle)(\rangle - q \langle \asymp \rangle \end{cases}$$



$$\boxed{J(D) := (-1)^{n^-} q^{n^+ - 2n^-} \langle D \rangle .}$$

0) $\langle \phi \rangle = 1.$

1) $\langle \bigcirc D \rangle = (q+q^{-1}) \langle D \rangle$

2) $\langle \times \rangle = \langle)(\rangle - q \langle \asymp \rangle$

e.g.

$$\langle \bigcirc \rangle \overset{(2)}{=} (q+q^{-1}) \langle \ \rangle \overset{(1)}{=} q+q^{-1}.$$

$$J(\bigcirc) = q+q^{-1}.$$

$$\begin{aligned}
\langle \infty \rangle &\overset{(3)}{=} \langle \bigcirc \bigcirc \rangle - q \langle \infty \rangle \\
&\overset{(2)}{=} (q+q^{-1})^2 \langle \ \rangle - q(q+q^{-1}) \langle \ \rangle \\
&\overset{(1)}{=} (q+q^{-1})^2 - q(q+q^{-1}) \\
&= (q+q^{-1})\{(q+q^{-1}) - q\} \\
&= q^{-1}(q+q^{-1}).
\end{aligned}$$

$$\begin{aligned}
J(\infty) &= (-1)^0 q^{1-0} \langle \infty \rangle \\
&= q+q^{-1}
\end{aligned}$$

Contents.

A categorification of the Jones polynomial (M. Khovanov, 2000)

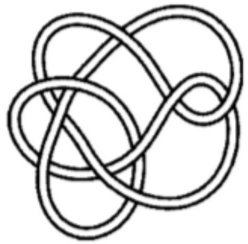
$$Kh(K) = \begin{array}{r|cccc}
-1 & \cdot & \cdot & \cdot & \mathbb{Q} \\
-3 & \cdot & \cdot & \cdot & \mathbb{Q} \\
-5 & \cdot & \mathbb{Q} & \cdot & \cdot \\
-7 & \cdot & \cdot & \cdot & \cdot \\
-9 & \mathbb{Q} & \cdot & \cdot & \cdot \\
\hline
 & -3 & -2 & -1 & 0
\end{array}$$

$K =$ 

$$J(K) = -q^{-9} + q^{-5} + q^{-3} + q^{-1}$$

Kh is strictly stronger than J!

5_1



10_{132}

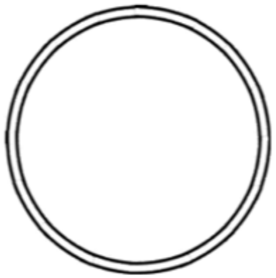
-3	.	.	.	.	.	$\mathbb{Q}$
-5	.	.	.	.	.	$\mathbb{Q}$
-7	.	.	.	$\mathbb{Q}$	.	.
-9	.	.	.	.	.	.
-11	.	$\mathbb{Q}$	$\mathbb{Q}$	.	.	.
-13	.	.	.	.	.	.
-15	$\mathbb{Q}$	.	.	.	.	.
	-5	-4	-3	-2	-1	0

$$\ncong$$

-1	.	.	.	.	.	.	$\mathbb{Q}$	$\mathbb{Q}$
-3	.	.	.	.	.	.	.	$\mathbb{Q}$
-5	.	.	.	.	$\mathbb{Q}$	$\mathbb{Q}^2$	.	.
-7	.	.	.	$\mathbb{Q}$	.	.	.	.
-9	.	.	.	$\mathbb{Q}$	$\mathbb{Q}$	.	.	.
-11	.	$\mathbb{Q}$	$\mathbb{Q}$	.	.	.	.	.
-13	.	.	.	.	.	.	.	.
-15	$\mathbb{Q}$	.	.	.	.	.	.	.
	-7	-6	-5	-4	-3	-2	-1	0

$$-q^{-15}+q^{-7}+q^{-5}+q^{-3} \quad = \quad -q^{-15}+q^{-7}+q^{-5}+q^{-3}$$

Kh detects the unknot!

$K =$  $\Longrightarrow$ / $\Longleftarrow$ $Kh(K) = \begin{array}{r|c} 1 & \mathbb{Q} \\ -1 & \mathbb{Q} \\ \hline & 0 \end{array}$

Construction

$A := \mathbb{Q}\{1, X\}$ ... 2-dim. $\mathbb{Q}$-vec. sp.

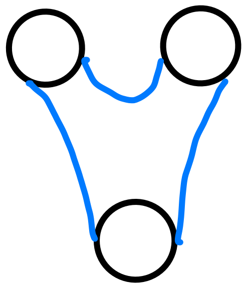
equipped with operations :

$$\left(\begin{array}{ll} m: A\otimes A \longrightarrow A, & \iota : \mathbb{Q} \longrightarrow A \\ \Delta : A \longrightarrow A\otimes A, & \varepsilon : A \longrightarrow \mathbb{Q} \end{array}\right)$$

... comm. Frobenius alg.

$\rightsquigarrow$ Gives rise to a (1+1)-TQFT :

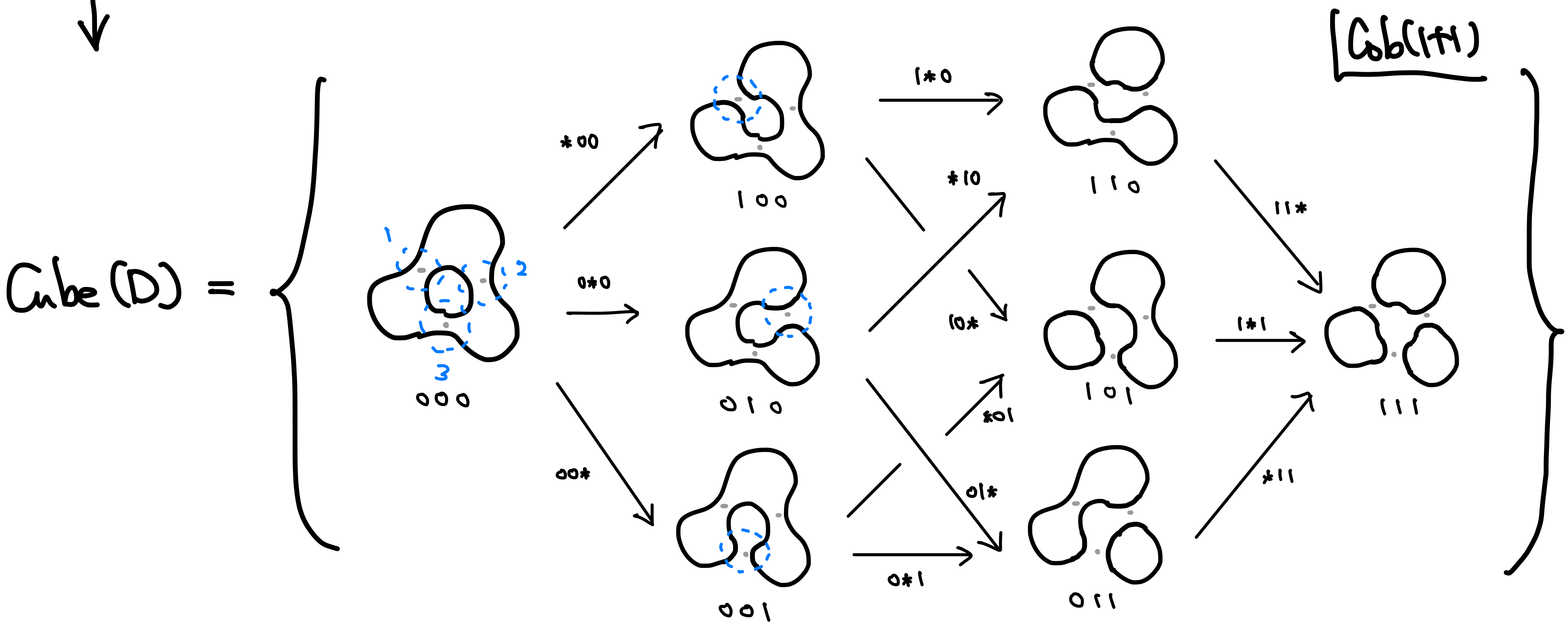
$$\mathcal{F}_A : \mathrm{Cob}(1+1) \longrightarrow \mathrm{Vect}_{\mathbb{Q}}$$



$$\begin{tikzcd} \longmapsto & A\otimes A \arrow[d, shift right, "m"'] \\ \longmapsto & A \arrow[u, dotted, shift right, "\Delta"'] \end{tikzcd}$$

etc.

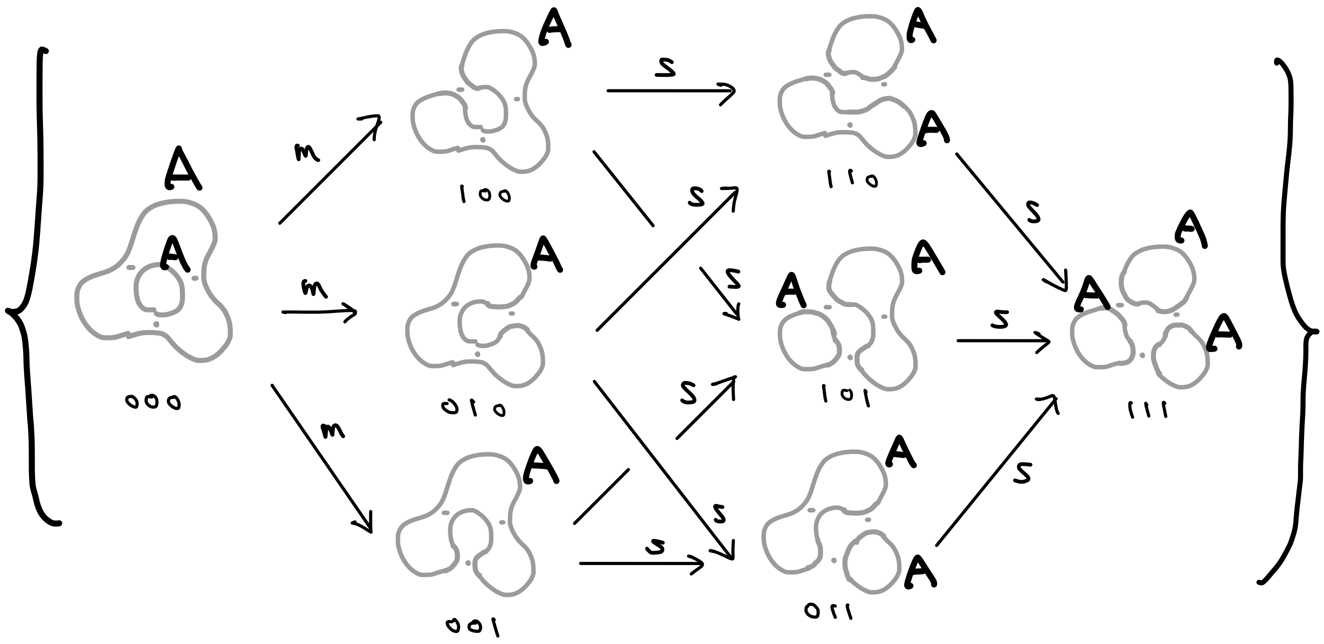
Step 1. Take cube of resolutions.



Step 2. Apply $\mathcal{F}_A$.

$\mathrm{Cob}(1+1)$

$\mathrm{Cube}(D) =$



m: merge

s: split

$$\Big\downarrow \mathcal{F}_A$$

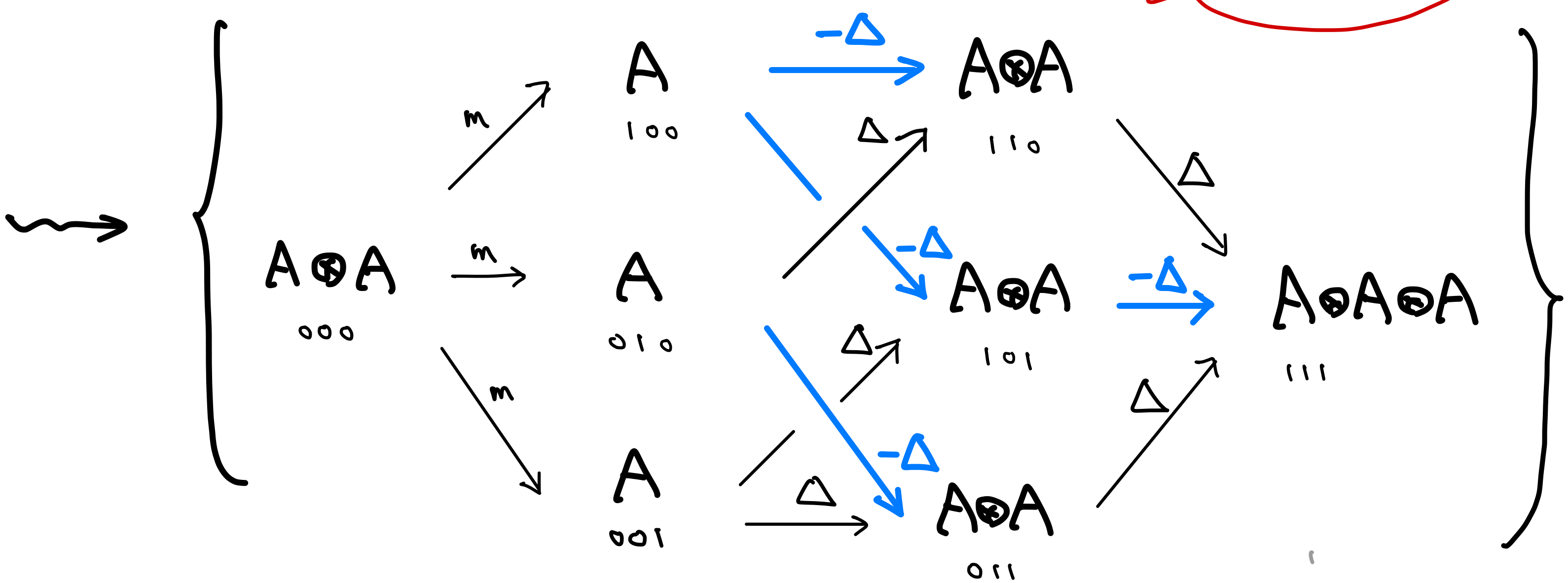
Mod_R

commutative cube

$$\mathcal{F}_A\,\mathrm{Cube}(D) = \left\{
\begin{tikzcd}
 & \underset{100}{A} \arrow[r, "\Delta"] \arrow[dr] & \underset{110}{A\otimes A} \arrow[dr, "\Delta"] & \\
\underset{000}{A\otimes A} \arrow[ur, "m"] \arrow[r, "m"] \arrow[dr, "m"] & \underset{010}{A} \arrow[ur, "\Delta"] \arrow[dr, "\Delta"] & \underset{101}{A\otimes A} \arrow[r, "\Delta"] & \underset{111}{A\otimes A\otimes A} \\
 & \underset{001}{A} \arrow[ur, "\Delta"] \arrow[r, "\Delta"] & \underset{011}{A\otimes A} \arrow[ur, "\Delta"] &
\end{tikzcd}
\right\}$$

Step 3. Anti-commutate & fold.

anti-commutative cube



deg d = (+1, 0)

⊕ ⊕

$d^2 = 0$

fold $\rightsquigarrow$ $\{\, C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} C^3 \,\}$

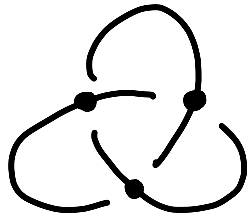
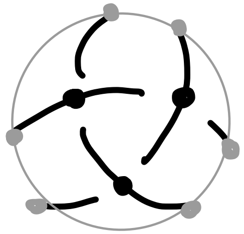
Step 4. Endow bigrading to get $CKh^{\cdot\cdot}(D)$. (done!)

Contents.

Bar-Natan's reformulation. (2004)

Motivations

- Understand the construction of Kh more geometrically.
- Generalize Kh to tangles.

knot :  $\rightsquigarrow$ tangle : 

Idea

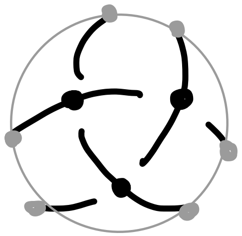
- Stay in the world of Topology!

  (Do not descend to the world of Algebra)

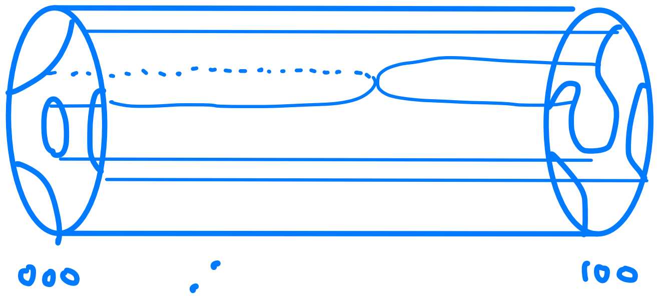
Construction

Step 1. Take cube of resolutions

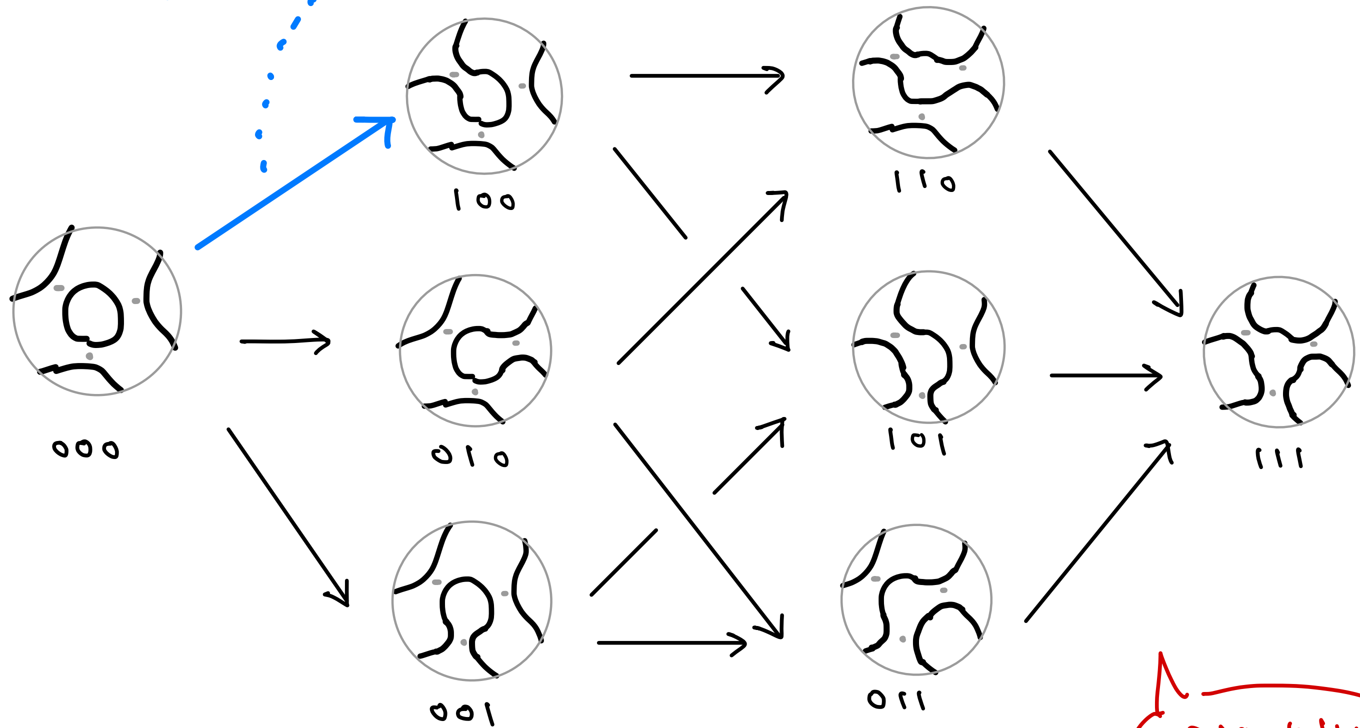
$T =$



edges are given by cobordisms.



$\mathrm{Cube}(T) = \Big\{$



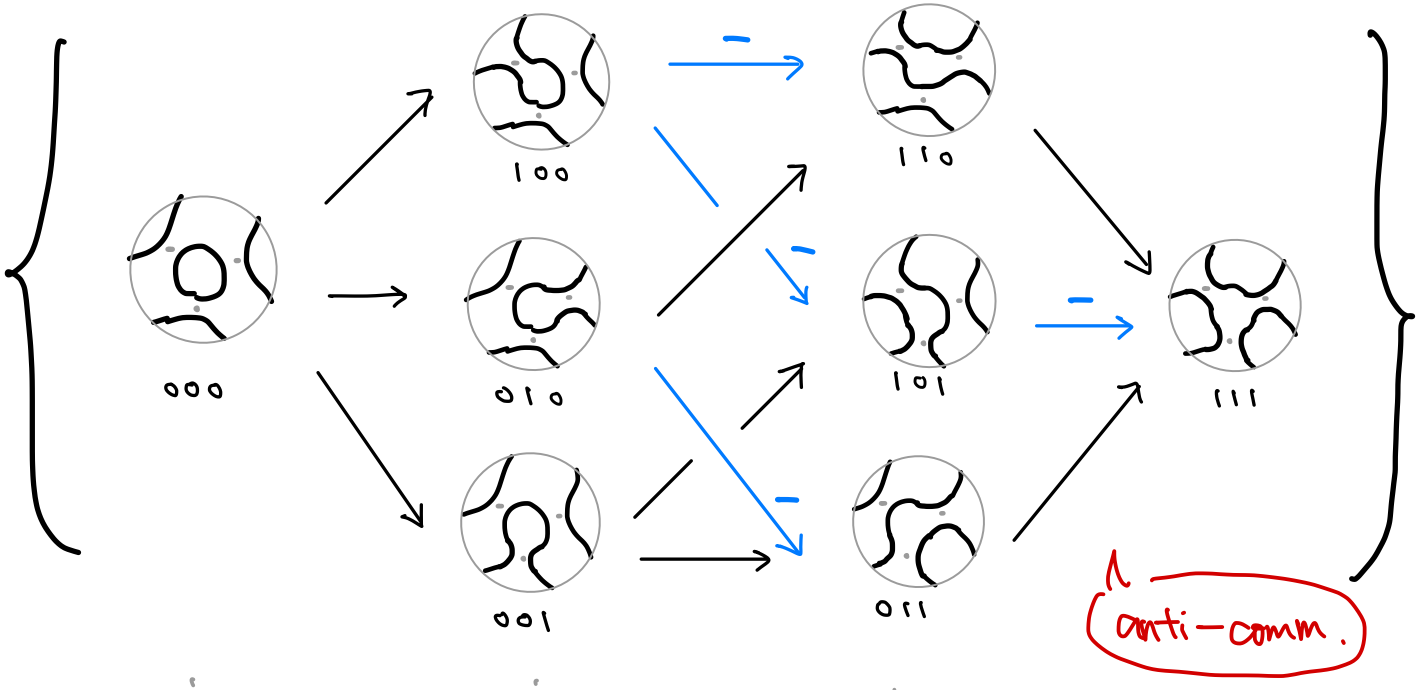
$\Big\}$

commutative

~~Step 2. Apply J'_A.~~

Step 2. Anti-commutate & fold (within the topological world!)

$\mathrm{Cube}(T) =$



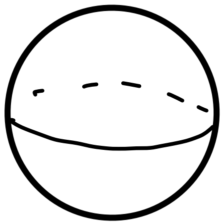
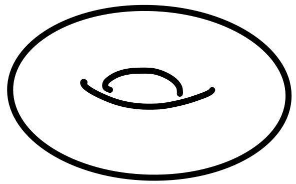
fold $\rightsquigarrow$

$$C(T) = \left\{ C^0 \xrightarrow{d} C^1 \xrightarrow{d} C^2 \xrightarrow{d} C^3 \right\}$$

$d^2 = 0$

Local relations.

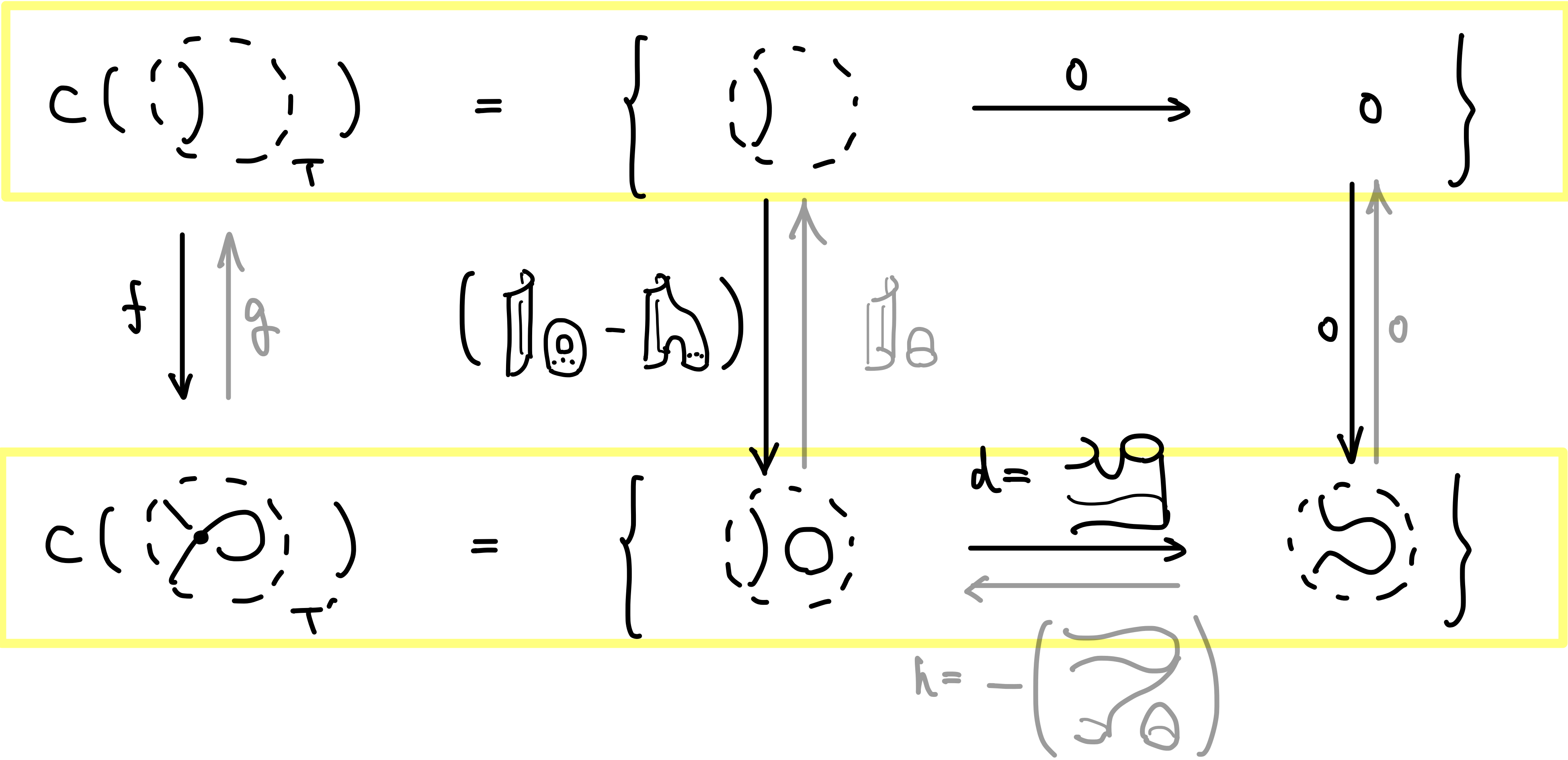
Endow the following local relations of surfaces:

(S)  $= 0$, (T)  $= 2$

(4Tu)  $+$  $=$  $+$ 

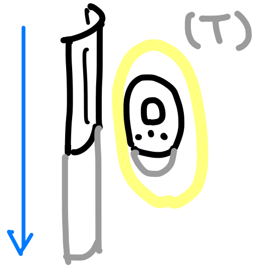
Using these local relations, one can prove that the "Khovanov homology for tangles" is a tangle invariant.

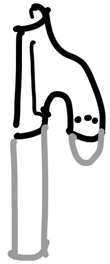
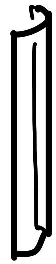
(Proof for RM1)

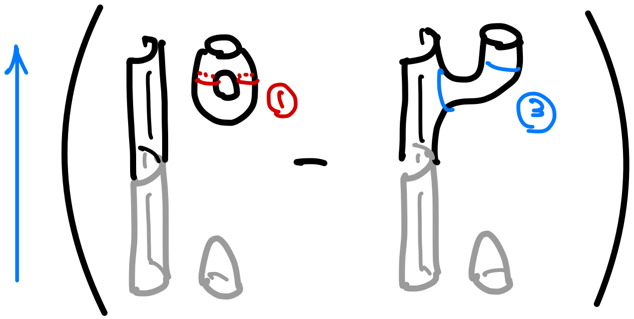


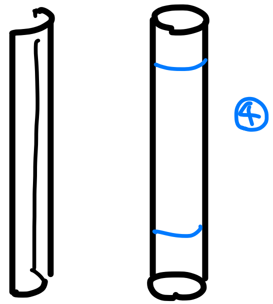
$$\Longrightarrow \begin{cases} g \circ f = 1 \\ f \circ g - 1 = h \circ d, \quad -1 = d \circ h. \end{cases}$$

(cont'd)

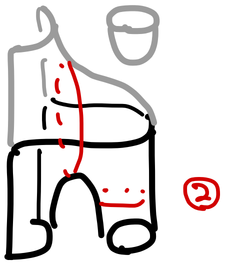
$g \circ f =$ 

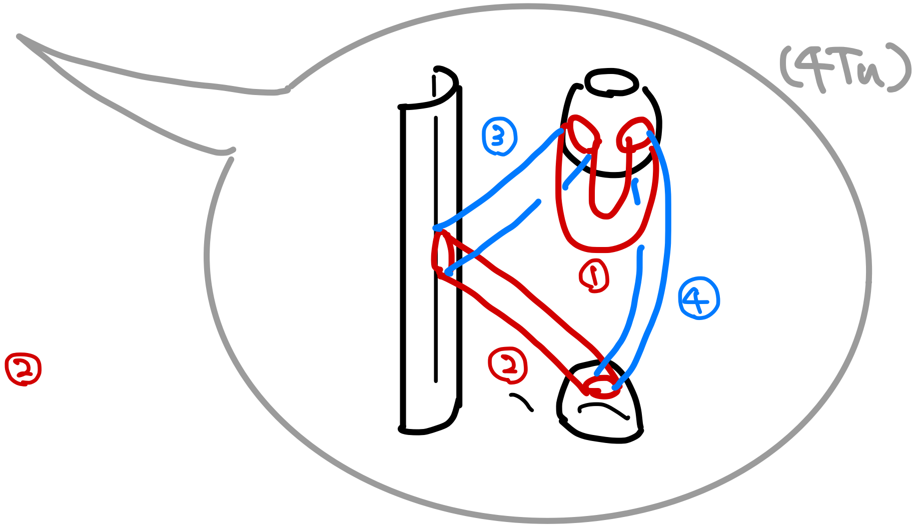
$-$  $=$  $= 1_{C(T)}$

$f \circ g - 1 =$ 

$-$ 

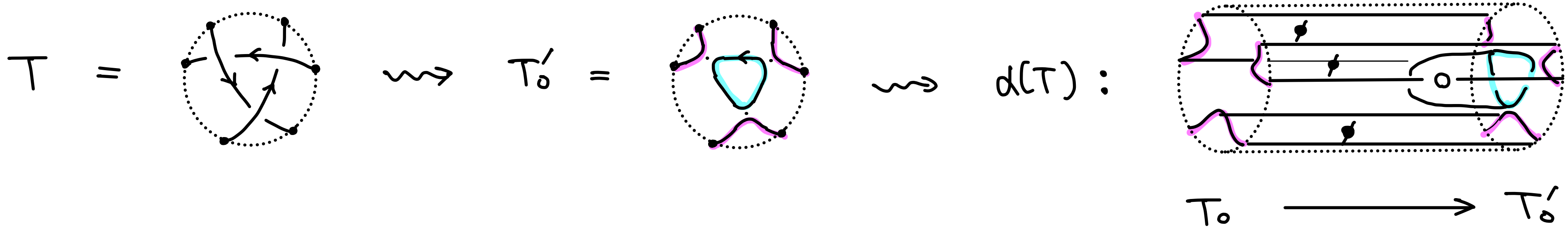
$\|$

$h \circ d = -$ 



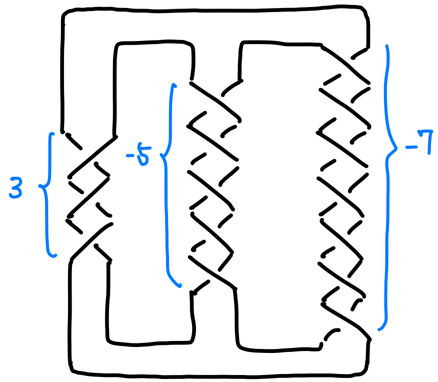
My recent work (w/ KeeTaek Kim) arxiv:2503.05414

Reformulate Rasmussen's s-invariant in the diagrammatic picture.



Application

Determined the s-invariants of all 3-strand pretzel knots.



P = P(3,-5,-7)

Thanks!