Higher Berry phases in quantum many-body systems

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Based on:

- ["Higher structures in matrix product states", Shuhei Ohyama and SR (23)]
- ["Higher Berry Phase from Projected Entangled Pair States in (2+1) dimensions", Shuhei Ohyama and SR (24)]
- ["Higher Berry Connection for Matrix Product States", Shuhei Ohyama and SR (24)]
- ["Multi wavefunction overlap and multi entropy for topological ground states in (2+1) dimensions", Bowei Liu, Junjia Zhang, Shuhei Ohyama, Yuya Kusuki, SR (24)]
- ["Higher Structures on Boundary Conformal Manifolds: Higher Berry Phase and Boundary Conformal Field Theory", Yichul Choi, Hyunsoo Ha, Dongyeob Kim, Yuya Kusuki, Shuhei Ohyama, SR (25)]

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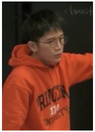
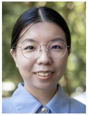
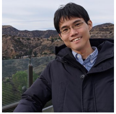
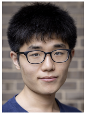


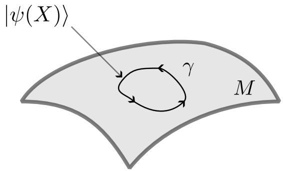
Thanks: Ken Shiozaki, Xueda Wen

Outline

- Introduction – regular v.s. higher Berry phases 6 pages
- (1+1)d tensor network (Matrix Product States, MPS) 11 pages
- (1+1)d continuum formulation (Boundary CFT, BCFT) 3 pages
- Summary and comments 3 pages
- (2+1)d tensor network (Projected Entangled Pair States, PEPS) 5 pages
- (2+1)d continuum formulation (Bulk-boundary correspondence) 2 pages

Regular Berry phase

- A quantum state $|\psi\rangle$ that depends smoothly on some parameter $X = (X^1, X^2, \cdots)$. X can change over some manifold M ("parameter space").



- The Berry phase is the phase accumulated by $|\psi\rangle$ as we change X smoothly along a loop γ

$$i \oint_\gamma dt \, \langle \psi(X(t))| \tfrac{d}{dt} |\psi(X(t))\rangle = i \oint_\gamma dt \, \underbrace{\langle \psi(X)| \frac{\partial}{\partial X^\mu} |\psi(X)\rangle}_{\equiv A_\mu(X)} \frac{dX^\mu}{dt}$$

$$= i \oint_\gamma A_\mu dX^\mu = i \oint_\gamma \mathcal{A}$$

Kinematics:

- The Berry phase governs the kinematics of X:

$$S[X] = \int dt \, \langle \psi(X) | i\partial_t - H(X) | \psi(X) \rangle$$

- $\mathcal{A}$ couples to a conserved current (particle trajectory) in parameter space,

$$i \oint_\gamma \mathcal{A} = i \int d^D X \, A_\mu j^\mu, \quad j^\mu(X) = \int dt \, \delta^D(X - X(t)) \dot{X}^\mu$$

Topology:

- A family of states $\{|\psi(X)\rangle\}$ over M is topologically non-trivial when the integral of the Berry curvature is non-zero:

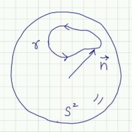
$$\int_M \mathcal{F} = \int_M d\mathcal{A} = 2\pi i \times \text{Integer}$$

- The integer (Chern number) measures the charge of a Dirac monopole in the parameter space.

Examples

- Single spin with $M = S^2$

$$|\psi(\vec{n})\rangle = e^{i\chi} \begin{pmatrix} e^{-i\phi/2}\cos\theta/2 \\ e^{+i\phi/2}\sin\theta/2 \end{pmatrix}$$

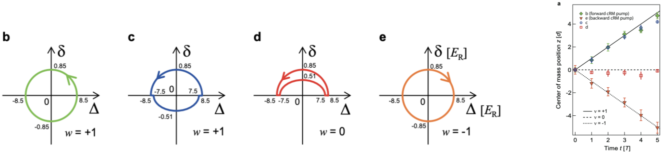


- Thouless pump [Thouless (83)]

$$H = \sum_i \left[-(J+\delta) f_i^\dagger d_i - (J-\delta) f_i^\dagger d_{i+1} + h.c. + \Delta(f_i^\dagger f_i - d_i^\dagger d_i) \right]$$

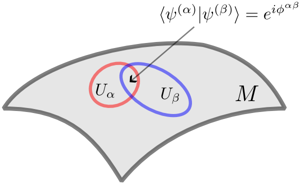
$$Ch = \int_0^T dt \int_{-\pi}^{\pi} dk\, \mathcal{F}(k,t)/2\pi i = \text{integer}$$

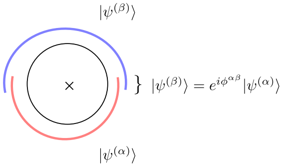
[Experiment; Nakajima et al (16)]



Wu-Yang's approach to magnetic monopole

- Consider patching M, so that for each patch, we can define a smooth gauge of wavefunction.





- Within the intersection $U_\alpha \cap U_\beta$, two wave functions, one from each patch, must be physically equivalent, and related by a gauge transformation

$$|\psi_\alpha\rangle = e^{i\phi_{\alpha\beta}} |\psi_\beta\rangle, \qquad e^{i\phi_{\alpha\beta}}: \text{ transition function}$$

- The winding number of $\phi_{\alpha\beta} \simeq$ Chern number

- The data $(\{U_\alpha\}, \{e^{i\phi_{\alpha\beta}}\})$ topologically defines a complex line bundle; "Chern class"

Higher Berry phase

- Higher Berry phase is a generalization of regular Berry phase for many-body systems or extended objects.

Berry connection $\mathcal{A} = A_\mu dX^\mu$	Higher Berry connection $\mathcal{B} = (1/2)B_{\mu\nu}dX^\mu dX^\nu$
Berry curvature $d\mathcal{A}$	Higher Berry curvature $d\mathcal{B}$
Particle	"String"' or (1+1)d many-body states
$\oint_\gamma \mathcal{A} = i\int d^DX\, A_\mu j^\mu$	$\oint_\gamma \mathcal{B} = i\int d^DX\, B_{\mu\nu} j^{\mu\nu}$
$\lvert\psi(X(t))\rangle$	$\lvert\psi[X(t,\sigma)]\rangle$

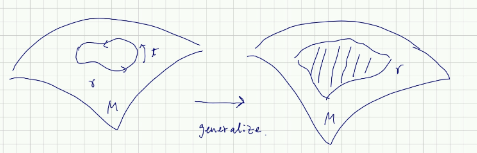
Some prior works: [Kitaev (2019); Kapustin-Spodyneiko (20); Kapustin-Sopenko (22); Hsin-Kapustin-Thorngren (20); Choi-Ohmori (22); Shiozaki (21); Wen-Qi-Beaudry-Moreno-Pflaum-Spiegel-Vishwanath-Hermele (21); Ohyama-Shiozaki-Sato (22); Ohyama-Terashima-Shiozaki (23); ...]

What we would expect...

- Roughly, we generalize $X(t) \to X(t,\sigma)$ and $|\psi(X(t))\rangle \to |\psi[X(t,\sigma)]\rangle$.
- Now we can define "string current", which generalizes "particle current":

$$j^\mu(X) = \int dt\, \delta^D(X - X(t)) \frac{dX^\mu}{dt}$$
$$\to j^{\mu\nu}(X) = \int dt d\sigma\, \delta^D(X - X(t)) \frac{1}{2}\left[\frac{\partial X^\mu}{\partial t}\frac{\partial X^\nu}{\partial \sigma} - \frac{\partial X^\nu}{\partial t}\frac{\partial X^\mu}{\partial \sigma}\right]$$

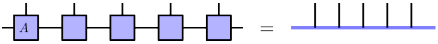
- Such string current can couple to a two-form gauge field $B_{\mu\nu}$:
- As before, we can consider $i\int_M B_{\mu\nu} j^{\mu\nu} = i\int_\gamma B_{\mu\nu} dX^\mu dX^\nu$.
There is a gauge invariance, $B_{\mu\nu} \to B_{\mu\nu} + \partial_\mu \xi_\nu - \partial_\nu \xi_\mu$.



Higher Berry phase for (1+1)d gapped states

- To proceed, we need to specify a class of states. I will focus on gapped, invertible (1+1)d states.

- Matrix product states (MPS):

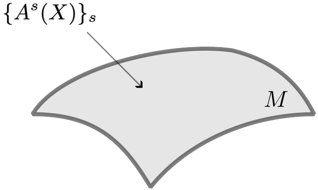


$$|\psi\rangle = \sum_{s_1, s_2, \cdots} \psi^{s_1 s_2 s_3 \cdots} |s_1, s_2, s_3 \cdots\rangle$$

$$\psi^{s_1 s_2 s_3 \cdots} = \sum_{i,j,k,l,m,n \cdots} A^{s_1}_{ij} A^{s_2}_{kl} A^{s_3}_{mn} \cdots$$

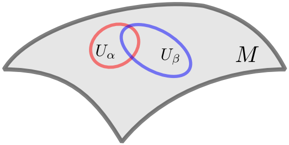
- Faithful representation of short-range entangled or invertible states in (1+1)d. E.g., symmetry-protected topological ground states

- A parameterized family of gapped short-range entangled states (MPS): $\{ \, | \, \psi[A] \, \rangle = | \, \psi[A(X)] \, \rangle \, \}$, $X \in M$.



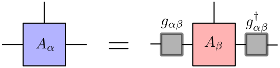
- Is there a topologically non-trivial family?
- Expect a generalization $\int_M d\mathcal{A} \longrightarrow \int_M d\mathcal{B}$, $H^2(M,\mathbb{Z}) \longrightarrow H^3(M,\mathbb{Z})$
- Here, we try to generalize Wu-Yang's description

- Now, as before, we patch M, $\{U_\alpha\}$.



- Over each patch U_α, MPS representation $\{A^s_\alpha(X)\}$ is smooth.
- When two patches intersect, two MPS represent the same physical state.
- The fundamental theorem then states that the MPS are related as

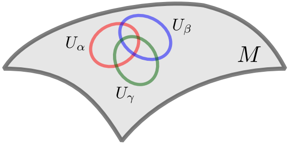
$$A^s_\alpha = g_{\alpha\beta} A^s_\beta g^\dagger_{\alpha\beta} e^{i\chi}$$



This relation should play a similar role as $|\psi_\alpha\rangle = e^{i\phi_{\alpha\beta}}|\psi_\beta\rangle$.
We call $g_{\alpha\beta}$ transition function.

Triple intersection

- When three patches intersect, on $U_\alpha \cap U_\beta \cap U_\gamma$, we can consider three transition functions, $g_{\alpha\beta}$, $g_{\beta\gamma}$, $g_{\gamma\delta}$ (with $g_{\beta\alpha} = g^\dagger_{\alpha\beta}$ etc.)



- For the case of regular Berry phases, we have $e^{i\phi_{\alpha\beta}}$, $e^{i\phi_{\beta\gamma}}$, $e^{i\phi_{\gamma\alpha}}$. They satisfy the cocycle condition $e^{i\phi_{\alpha\gamma}} = e^{i\phi_{\alpha\beta}} e^{i\phi_{\beta\gamma}}$ since we can "relate" $|\psi_\gamma\rangle$ and $|\psi_\alpha\rangle$ in two different ways:

$$|\psi_\alpha\rangle = e^{i\phi_{\alpha\gamma}}|\psi_\gamma\rangle \quad : \text{ direct way}$$
$$|\psi_\alpha\rangle = e^{i\phi_{\alpha\beta}}|\psi_\beta\rangle = e^{i\phi_{\alpha\beta}} e^{i\phi_{\beta\gamma}}|\psi_\gamma\rangle \quad : \text{ indirect way}$$

Triple intersection

- Let us repeat this argument, and relate A^s_γ and A^s_α in two different ways:

$$A^s_\alpha = g_{\alpha\gamma} A^s_\gamma g^\dagger_{\alpha\gamma} \quad : \text{direct way}$$
$$A^s_\alpha = g_{\alpha\beta} A^s_\beta g^\dagger_{\alpha\beta} = g_{\alpha\beta} g_{\beta\gamma} A^s_\gamma g^\dagger_{\beta\gamma} g^\dagger_{\alpha\beta} \quad : \text{indirect way}$$

- Once again, we demand the consistency; however, there is a $U(1)$ ambiguity,

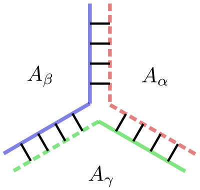
$$g_{\alpha\beta} g_{\beta\gamma} = g_{\alpha\gamma} \times c_{\alpha\beta\gamma}$$

  "Ambiguity of ambiguity" (g is an ambiguity of A, c is an ambiguity of g)

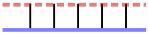
- Just like $\{e^{i\phi_{\alpha\beta}}\}$ defines an element in $H^2(M, \mathbb{Z})$. Viz, Chern class, $\{c_{\alpha\beta\gamma}\}$ defines an element in $H^3(M, \mathbb{Z})$. This class is called the Dixmir-Douady class, which is a topological invariant.

- The transition function $e^{i\phi_{\alpha\beta}}$ can be "extracted" from the inner product $\langle\psi_\alpha|\psi_\beta\rangle$ – This is the work of Wu-Yang relating physics (QM with monopole) and mathematics (complex line bundle).
  Is there any physical quantity related to $c_{\alpha\beta\gamma}$?

- For this purpose, we generalize the concept of inner product in quantum mechanics to triple inner product,

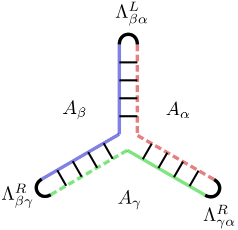


- Recall that the regular inner product:



  takes two quantum states (MPS) and spits out one complex number.
  The triple inner product takes three states and spits one number.

- In the thermodynamic limit, this diagram is evaluated as



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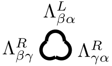
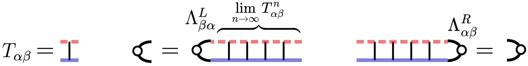


$$= \mathrm{tr}\left(\Lambda^L_{\beta\alpha}\Lambda^R_{\beta\gamma}\Lambda^R_{\gamma\alpha}\right) = \mathrm{tr}\left(\Lambda^L_{\alpha}\hat{g}_{\alpha\beta}1_n\hat{g}_{\beta\gamma}1_n\hat{g}_{\gamma\alpha}\right)$$

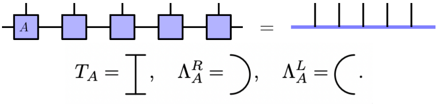
$$= \mathrm{tr}\,(\Lambda^L_\alpha)c_{\alpha\beta\gamma}$$

- Transfer matrix:



Curvature and connection

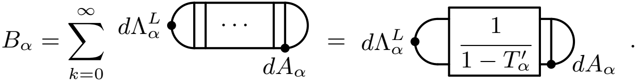
- So far we have developed an analogue of Wu-Yang's description.
- Alternative perspective: Berry's connection: $A = \langle\psi|d\psi\rangle$ and curvature $F = dA$. The topological invariant as the integral of the curvature.
- Can we write a down 2-form connection using wavefunctions (tensor network)?



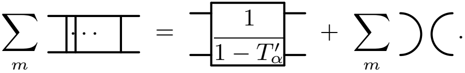
- [Kapustin-Spodyneiko (20)]
- [Kapustin-Sopenko (22)] [Artymowicz-Kapustin-Sopenko (23)]
- [Shiozaki-Heinsdorf-Ohyama (23)] Discretized version of the 2-form connection.
- [Sommer-Wen-Vishwanath (24)] [Ohyama-SR (24)]
- . . .

[Sommer-Wen-Vishwanath (24)] [Ohyama-SR (24)]

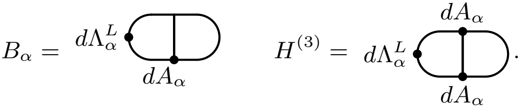
- 2-form connection:



by summing over "ladder diagrams":



- For fixed point MPS:



Example; $M = S^3$

[Wen-Qi-Beaudry-Moreno-Pflaum-Spiegel-Vishwanath-Hermele (21)]

Dimerized spin chain

$$H(\vec{w}) = H^{\text{on-site}}(\mathbf{w}) + H^{\text{odd}}(w_4) + H^{\text{even}}(w_4)$$

with parameter $M = S^3 = \{\vec{w} = (\mathbf{w}, w_4) \,|\, \sum_{\mu=1}^{4} w_\mu^2 = 1\}$

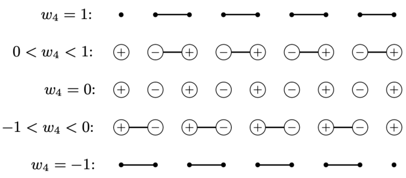
$$H^{\text{on-site}}(\mathbf{w}) = \sum_p (-1)^p \mathbf{w} \cdot \boldsymbol{\sigma}_p \,,$$

$$H^{\text{odd}}(w_4) = \sum_{p:\text{odd}} g^{\mathrm{N}}(\vec{w}) \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_{p+1} \,,$$

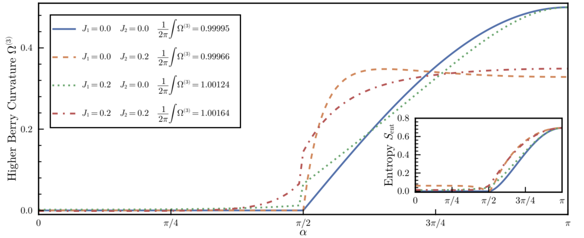
$$H^{\text{even}}(w_4) = \sum_{p:\text{even}} g^{\mathrm{S}}(\vec{w}) \boldsymbol{\sigma}_p \cdot \boldsymbol{\sigma}_{p+1} \,.$$

$$g^{\mathrm{N}}(\vec{w}) = \begin{cases} w_4 & (0 \le w_4 \le 1)\,, \\ 0 & (-1 \le w_4 \le 0)\,, \end{cases}$$

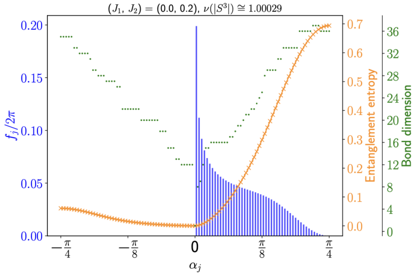
$$g^{\mathrm{S}}(\vec{w}) = \begin{cases} 0 & (0 \le w_4 \le 1)\,, \\ -w_4 & (-1 \le w_4 \le 0)\,. \end{cases}$$

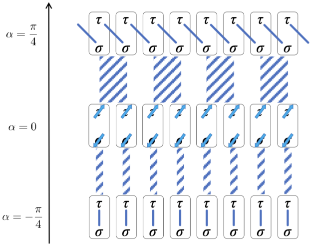


[Sommer-Wen-Vishwanath (24)]



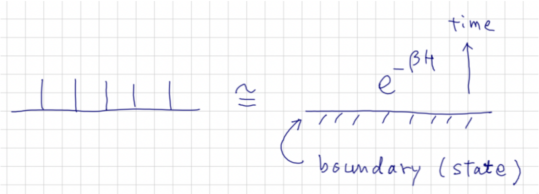
[Shiozaki-Heinsdorf-Ohyama (23)]



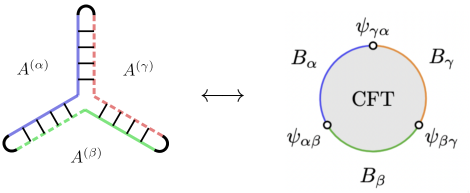


Continuum formulation using BCFT

- So far, we discussed discrete (lattice) formalism; How about continuum systems? Do we need tensor network representations?
- We have developed a formalism using boundary CFT (BCFT) Why is (B)CFT relevant to gapped ground states?
- Boundary states in BCFT represent boundary conditions by exchanging the role of space and time
- Boundary states (with suitable smearing) can be used to model (1+1)d gapped ground states [Qi-Katsura-Ludwig (12); Miyaji-SR-Takayanagi-Wen(14); Cardy (17);Cho-Shiozaki-SR-Ludwig(17)]



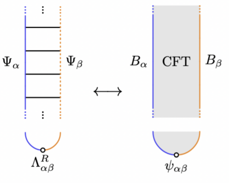
- Triple inner product can be translated into disk partition function with three boundary conditions $B_\alpha, B_\beta, B_\gamma$



- Between different boundary conditions, we have boundary-condition-changing (bcc) operators $\psi_{\alpha\beta}$
- Higher Berry phase = the phase of three-point function of bcc operators.

Comments

- Bcc operators can be thought of as the fixed point of mixed transfer matrix



- Ideal entanglement spectrum in gapped integrable spin chains [Date-Jimbo-Miwa-Okado (87)]

- We can also make a smooth modulation $B_\alpha \to B(x)$. Connection to loop space connection (Wess-Zumino term) [C.f. Mickelson (87), Stone (89), Iso-Itoi-Mukaida (90), ...]

Summary and further comments

- For a parameterized family of gapped invertible states (MPS, boundary states), we developed a framework to calculate the higher Berry phase and topological invariant.
- This is in parallel with the works of Wu-Yang and Berry.
- What was essential is the gauge redundancy of MPS, going beyond the regular phase ambiguity of quantum states.
- Comment 1: connection to string field theory
- Comment 2: higher-dimensional generalization
  - Tensor network formulation (Projected entangled pair states)
  - Continuum formulation using the bulk-boundary correspondence
- Comment 3: connection to multipartite entanglement (multi-entropy)

String field theory and $*$ product

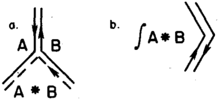
- "Aharonov-Bohm phase" of MPS: 2-form gauge field
- Possible because of "internal structure" of MPS, can go beyond single-particles
- 2-form gauge field couples naturally to 1d extended objects; strings
- No fundamental string in condensed matter physics, but we could have emergent ones.
- More (formal) direct link with string theory (string field theory):

NON-COMMUTATIVE GEOMETRY AND STRING FIELD THEORY

Edward WITTEN*

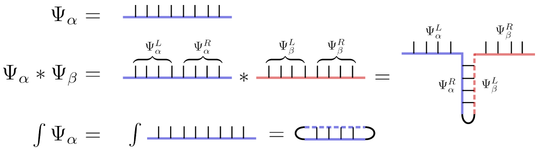
Joseph Henry Laboratories, Princeton University, Princeton, New Jersey 08540, USA

Received 2 December 1985

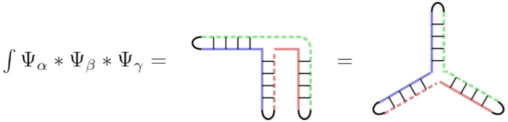


String $\star$ product and triple inner product

- Following [Witten (85)], we can introduce "$\star$" product and "integration" for MPSs

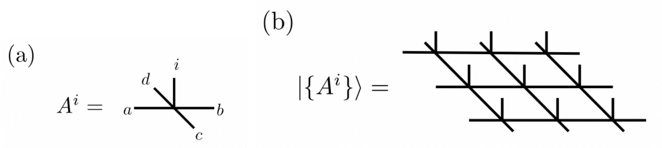


- Triple inner product is given by "string field theory vertex"

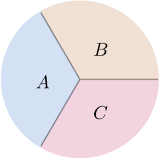


Generalizations to higher dimensions

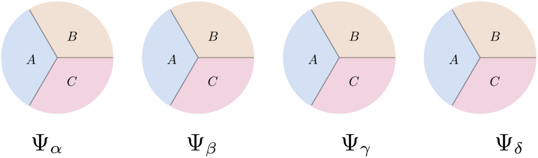
- Generalizations to higher dimensions; from stings to membranes
- We consider (2+1)d gapped state (invertible or topologically-ordered). They can be represented by 2d tensor network, e.g., Projected Entangled Pair States (PEPS) of some sort.
- PEPS can represent a class of invertible states, e.g., SPT ground states



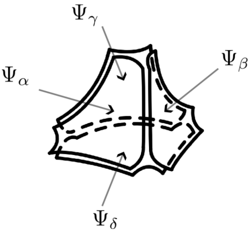
- We first divide the 2d space (infinite plane) into three regions, A, B, C:



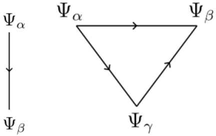
- Consider four states, Ψ_α, Ψ_β, Ψ_γ, Ψ_δ. Each of them is tripartitioned.
- We can now "connect" or "contract" these states with each other.



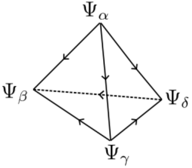
- It's a bit complicated to draw, but the result of the contraction looks like:



- A simplified notation: the triple inner product for three bipartite states, $|\Psi\rangle = \sum \psi_{ij} |i\rangle_A |j\rangle_B$.
- In this notation, the regular inner product is simply a line segment, triple innner product is a triangle:



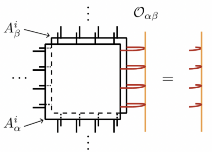
- The quadruple inner product is defined for four tripartite states, $|\Psi\rangle = \sum_{ijk} \psi_{ijk} |i\rangle_A |j\rangle_B |k\rangle$. So, each wave function may be viewed as a trivalent vertex.



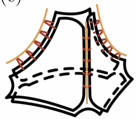
Quadruple inner product

- Let's "evaluate" the quadruple product.
- As in the (1+1)d case, it is important to think what happens at infinities.
- For example, two states Ψ_α and Ψ_β "meet": Following the (1+1)d case, we need to contract the boundary indices. So we use Matrix Product Operators (MPO) or Matrix Product Unitaries (MPU).
- This is not the end of the story. We also have corners, where three states meet.
- At the corners, three MPU must meet. So, we also need a tensor to connect different MPUs:

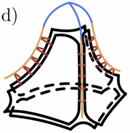
(c)



(c)

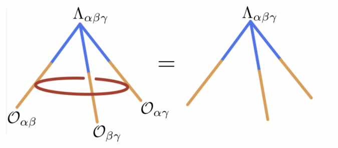


(d)

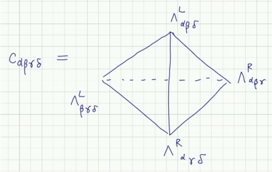


Quadruple inner product

- The 3-leg tensor should satisfy the fixed point condition.



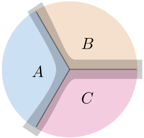
- In the end, the quadruple inner product reduces to a network (tetrahedron) formed by 3-leg tensors (and MPU):

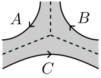


- This is analogous to (1+1)d case where the triple inner product reduces to a triangle of fixed point tensors.
- Concrete implementation and calculations using semi injective PEPS [Molnar-Ge-Schuch-Cirac (18)]
  - Examples: (2+1)d SPT and others

Bulk-boundary approach to multi overlap

- Bulk-boundary correspondence: [Bowei Liu-Kusuki-Ohyama-SR (24)] [Yuhan Liu-Kusuki-Sohal-Kudler-Flam-SR (23)]
- Multi wavefunction overlap of (2+1)d gapped invertible ground states can be represented by a CFT partition function in (1+1)d

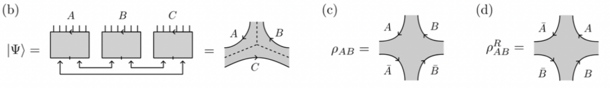




- Similar approach was developed for topological entanglement entropy (for bipartition setting) and related quantities

Edge theory approach to quadruple inner product

- Rewrite multi overlap as $\mathrm{Tr}\left[(\rho_{AB}^R)^\dagger \rho_{AB}\right]$ where ρ_{AB}^R is realignment of ρ_{AB}.
- For a density matrix $\rho_{AB} = \sum_{i,k=1}^{\dim \mathcal{H}_A} \sum_{j,l=1}^{\dim \mathcal{H}_B} \rho_{ij,kl} |ij\rangle\langle kl|$ on $\mathcal{H}_A \otimes \mathcal{H}_B$, $\rho_{AB}^R = \sum_{i,k=1}^{\dim \mathcal{H}_A} \sum_{j,l=1}^{\dim \mathcal{H}_B} \rho_{ij,kl} |ik\rangle\langle jl|$.
- ρ_{AB} obtained after tracing C can be represented as:



- Multi overlap can be represented by a CFT partition function in (1+1)d: