Spacetime and Matter from Matrices -Emergence of General Relativity in IIB Matrix Model

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Matters to Spacetime: Symmetries & Geometry

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Emergence of GR from IIB Matrix Model

1. IIB matrix model

Ishibashi-HK-Kitazawa-Tsuchiya

$$S_{\mathrm{IIB}} = \alpha\, Tr\left(-\frac{1}{4}\left[A_\mu, A_\nu\right]^2 - \frac{1}{2}\bar{\psi}\Gamma^\mu\left[A_\mu, \psi\right]\right)$$

A_μ and ψ are Hermitian operators on a vector space V.

$V = \mathbb{C}^N$

$\Rightarrow$ A_μ and ψ are $N \times N$ matrices

$(A_\mu)_{ij}$, $(\psi_s)_{ij}$

$V = \{\, \varphi(x) : \mathbb{R}^d \to \mathbb{C}\}$ the space of functions on $\mathbb{R}^d$

$\Rightarrow$ A_μ and ψ are bilocal fields

$$A_\mu(x, y) = \langle x|A_\mu|y\rangle\,,\ \psi_s(x.y) = \langle x|\psi_s|y\rangle$$

In the case $V = \{\, \varphi(x) : \mathbb{R}^d \to \mathbb{C}\}$, the action becomes

$$S = -\frac{\alpha}{2}\int d^d x d^d y d^d z d^d w$$
$$\{A_\mu(x,y)A_\nu(y,z)A_\mu(z,w)A_\nu(w,x)$$
$$-A_\mu(x,y)A_\nu(y,z)A_\nu(z,w)A_\mu(w,x)\}$$
$$+\int d^d x d^d y d^d z$$
$$\{\bar{\psi}(x,y)\Gamma^\mu A_\mu(y,z)\psi(z,x)$$
$$-\bar{\psi}(x,y)\Gamma^\mu \psi(y,z)A_\mu(z,x)\}.$$

Diffeomorphism invariance

This is invariant under the diffeomorphisms on $\mathbb{R}^d$:

$$A_\mu(x,y) \mapsto J(x)^{\frac{1}{2}} J(y)^{\frac{1}{2}} A_\mu(\varphi(x),\varphi(y)),$$

$$\psi(x,y) \mapsto J(x)^{\frac{1}{2}} J(y)^{\frac{1}{2}} \psi(\varphi(x),\varphi(y)),$$

where $J(x) = \det(\partial_\alpha \varphi^\beta(x))$.

It is because

$$\int d^d y \cdots \phi(x,y)\phi(y,z) \cdots$$
$$\mapsto \int d^d y \cdots \phi\big(\varphi(x),\varphi(y)\big)\, J(y)\, \phi\big(\varphi(y),\varphi(z)\big) \cdots$$
$$= \int d^d y' \cdots \phi(x',y')\phi(y',z') \cdots .$$

This a small subset of the $\mathrm{SU}(N)$ invariance:

$$|x\rangle \mapsto J(x)^{\frac{1}{2}} |\varphi(x)\rangle \quad \sim \text{ permutation of } |i\rangle$$

IIB matrix model as pre-geometric theory

We can view this action as a pre-geometric theory, i.e., a theory that, when a certain background is introduced, becomes an ordinary field theory on curved spacetime.

It is natural to expect that the low-energy effective theory of this action is a field theory involving gravity.

Roughly speaking, this is true, but we have to be careful about unitarity (or positivity).

This is because the vector and spinor indices of A and ψ are like Lorentz indices which are not transformed under the diffeomorphisms.

$$A_\mu(x,y) \mapsto J(x)^{\frac{1}{2}} J(y)^{\frac{1}{2}} A_\mu(\varphi(x),\varphi(y))$$

$$\psi_s(x,y) \mapsto J(x)^{\frac{1}{2}} J(y)^{\frac{1}{2}} \psi_s(\varphi(x),\varphi(y))$$

However, the IIB matrix model has no built-in local Lorentz invariance.

Therefore the situation is similar to an action written in terms of the vierbein $e_a^{\ \nu}$ but without local Lorentz symmetry.

In general, such theory is not unitary because the negative norm states $e_a^{\ 0}$ cannot be removed.

Constraints to guarantee unitarity

In fact, as we will see, one simple way to guarantee unitarity is to impose the following constraints on the bilocal fields.

$$\int d^d y\, A_\mu(x,y) = 0, \quad \int d^d y\, \psi_s(x,y) = 0\,.$$

If we do so, the symmetry is reduced from the diffeomorphisms to the volume preserving diffeomorphisms (VPDs)

$$J(x) = \det\left(\partial_\alpha \varphi^\beta(x)\right) = 1\,,$$

because these constraints are invariant under the diffeomorphisms only when $J = 1$.

Meaning of the constraints

$\leftarrow \phi(x,y) = \langle x|\phi|y\rangle$

$$\int d^d y\, \phi(x,y) = 0 \;\Leftrightarrow\; \phi|k=0\rangle = 0$$

Here $|k\rangle$ is the momentum eigenstates:

$$|k\rangle = \int d^d x \exp(i\, k\cdot x)\, |x\rangle$$
$$\Rightarrow\; |k=0\rangle = \int d^d x\, |x\rangle\,.$$

These constraints correspond to remove $|k=0\rangle$ from the Hilbert space $V = \{\varphi :\ \mathbb{R}^d \to \mathbb{C}\}$.

In other words, we replace

$$V = \langle |k\rangle \rangle_{k\in\mathbb{R}^d} \text{ with } V' = \langle |k\rangle \rangle_{k\in\mathbb{R}^d - \{0\}}\,.$$

This change seems very minor, but it imposes some nontrivial constraints.

formal Taylor expansion with respect to $\hat{p}_\mu$

$=$ differential operator acting on V

$$\phi = a(\hat{x}) + \frac{1}{2}\left(\hat{p}_\mu a^\mu(\hat{x}) + a^\mu(\hat{x})\hat{p}_\mu\right) + \frac{1}{2}\left(\hat{p}_\mu\hat{p}_\nu a^{\mu\nu}(\hat{x}) + a^{\mu\nu}(\hat{x})\hat{p}_\mu\hat{p}_\nu\right) + \cdots$$

$$= a(x) + \frac{i}{2}\left(\partial_\mu a^\mu(x) + a^\mu(x)\partial_\mu\right) + \frac{i^2}{2}\left(\partial_\mu\partial_\nu a^{\mu\nu}(x) + a^{\mu\nu}(x)\partial_\mu\partial_\nu\right) + \cdots$$

$$\phi\,|k=0\rangle = 0 \;\Leftrightarrow\; \begin{cases} a(x) - \frac{1}{2}\,\partial_\mu\partial_\nu a^{\mu\nu}(x) + \cdots = 0, \\ \partial_\mu a^\mu(x) - \partial_\mu\partial_\nu\partial_\lambda a^{\mu\nu\lambda}(x) + \cdots = 0\,. \end{cases}$$

When we expand the bilocal fields, we have infinitely many tensor fields:

$$A_\mu = a_\mu(x) + \frac{i}{2}\left(\partial_\nu a_\mu^{\ \nu}(x) + a_\mu^{\ \nu}(x)\partial_\nu\right) + \frac{i^2}{2}\left(\partial_\nu\partial_\lambda a_\mu^{\ \nu\lambda}(x) + a_\mu^{\ \nu\lambda}(x)\partial_\nu\partial_\lambda\right) + \cdots .$$

We expect that the higher rank tensors become massive while the lower rank tensors remain massless because of the symmetry.
In general, we need gauge symmetry or some constraints to eliminate negative norm states for massless fields.

We will show that the constraints indeed fulfill this role:

$$\partial_\nu a_\mu^{\ \nu}(x) + \cdots = 0 .$$

2. Commuting background

In the case $V = \mathbb{C}^N$, one of the simplest classical solutions is given by the diagonal matrices:

$$A_\mu^{(0)} = \begin{cases} P_\mu \ (\mu = 1 \sim d), \ \ \left(P_\mu\right)_{ij} = p_\mu^i \delta_{ij} \ \text{(diagonal)}, \\ 0 \ (\mu = d+1 \sim 10) \end{cases}$$

$$\psi^{(0)} = 0.$$

Quenched reduced model

Suppose

(1) $p_\mu^i \ (i = 1, \cdots, N)$ are uniformly distributed in $\mathbb{R}^d$,

(2) $\left(A_\mu\right)_{ii} = p_\mu^i$ and $(\psi)_{ii} = 0$ are fixed,

(3) $N \to \infty$ with αN fixed.

Then, the theory is equivalent to d-dim large-N YM.

Corresponding solution in $V = \{\, \varphi(x) : \mathbb{R}^d \to \mathbb{C} \}$

Using the momentum representation

$$|k\rangle = \int d^d x \, \exp(ik \cdot x) |x\rangle,$$

we introduce the identification

$$|i\rangle \in \mathbb{C}^N \leftrightarrow \; |k = p^i\rangle \in \{\, \varphi(x) : \mathbb{R}^d \to \mathbb{C} \} \,.$$

Then the diagonal solution

$$\left\langle i \middle| A_\mu^{(0)} \middle| j \right\rangle = \begin{cases} p_\mu^i \, \delta_{ij} & (\mu = 1 \sim d) \\ 0 & (\mu = d+1 \sim 10) \end{cases}, \qquad \langle i | \psi^{(0)} | j \rangle = 0$$

becomes

$$\left\langle k \middle| A_\mu^{(0)} \middle| k' \right\rangle = \begin{cases} k_\mu \delta(k - k') & (\mu = 1 \sim d) \\ 0 & (\mu = d+1 \sim 10) \end{cases}, \quad \langle k | \psi^{(0)} | k' \rangle = 0.$$

By introducing $\hat{x}^\mu$ and $\hat{p}_\mu$ as the ordinary QM

$$[\hat{x}^\mu, \hat{p}_\nu] = i\delta^\mu{}_\nu,$$

we can express

$$A_\mu^{(0)} = \begin{cases} \hat{p}_\mu \ (\mu = 1 \sim d) \\ 0 \quad (\mu = d+1 \sim 10) \end{cases}$$

$$\psi^{(0)} = 0\,.$$

In terms of the coordinate representation,

$$\left\langle x \middle| A_\mu^{(0)} \middle| x' \right\rangle = \begin{cases} -i\,\partial_\mu \delta(x - x') & (\mu = 1 \sim d) \\ 0 & (\mu = d+1 \sim 10) \end{cases}$$

$$\left\langle x \middle| \psi^{(0)} \middle| x' \right\rangle = 0.$$

$A_\mu^{(0)}$ = Flat spacetime background

Commutator with $A_\mu^{(0)}$ is the derivative with respect to the center of mass coordinate of the bilocal fields.

In fact,

$$\left\langle x\left|\left[A_\mu^{(0)},\phi\right]\right|x'\right\rangle = \langle x|[\hat{p}_\mu,\phi]|x'\rangle = -i\left(\frac{\partial}{\partial x^\mu}+\frac{\partial}{\partial x'^\mu}\right)\langle x|\phi|x'\rangle.$$

$$= -i\frac{\partial}{\partial X^\mu}\phi\left(X+\frac{\xi}{2},X-\frac{\xi}{2}\right),$$

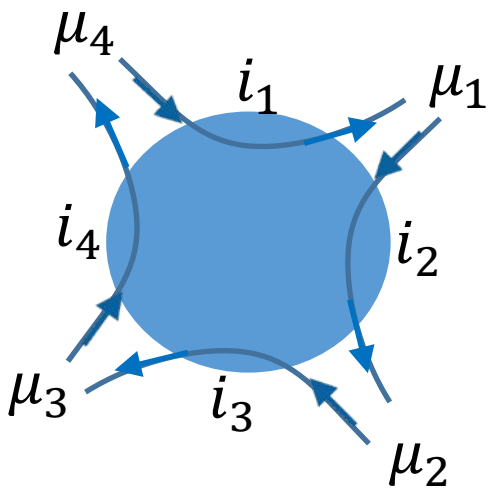
where X and ξ are the CM and the relative coordinates

$$X=\frac{1}{2}(x+x'),\ \xi=x-x'.$$

We can regard $A_\mu^{(0)}$ as a flat space background of the pre-geometric action.

3. Quenched reduced model and bilocal field

If the diagonal elements of A_μ are distributed uniformly in $\mathbb{R}^d$ and only planar diagrams are considered, the YM-type matrix model is equivalent to d-dim field theory:



$$\left\langle (A_{\mu_1})_{i_1 i_2} \cdots (A_{\mu_L})_{i_L i_1} \right\rangle_{\mathrm{MM}} = \left\langle \tilde{A}_{\mu_1}(p^{i_1 i_2}) \cdots \tilde{A}_{\mu_L}(p^{i_L i_1}) \right\rangle_{\mathrm{FT}}$$

$\mathrm{tr}(t^{a_1} \cdots t^{a_L}) \times$

where $p_\mu^{ij} = p_\mu^i - p_\mu^j$.

Parisi, Gross-Kitazawa,
Bhanot-Heller-Neuberger

In terms of bilocal field in momentum representation

$$\left\langle \langle k_1 | A_{\mu_1} | k_2 \rangle \cdots \langle k_L | A_{\mu_L} | k_1 \rangle \right\rangle_{\mathrm{MM}}$$
$$= \left\langle \tilde{A}_{\mu_1}(k_1 - k_2) \cdots \tilde{A}_{\mu_L}(k_L - k_1) \right\rangle_{\mathrm{FT}}$$

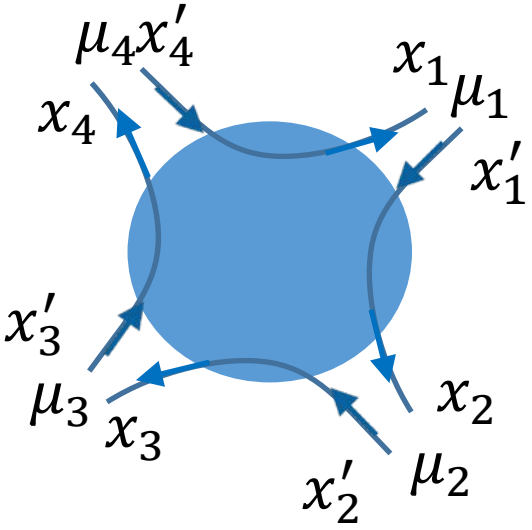
In particular, 2 pt. Green's function is

$$\left\langle \langle k_1|A_{\mu_1}|k_1'\rangle \langle k_2|A_{\mu_2}|k_2'\rangle \right\rangle_{\mathrm{MM}}$$
$$= \delta(k_1' - k_2)\delta(k_2' - k_1)\left\langle \tilde{A}_{\mu_1}(k_1 - k_1')\, \tilde{A}_{\mu_2}(k_2 - k_2') \right\rangle_{\mathrm{FT}}$$

In the coordinate representation

$$\left\langle \langle x_1|A_{\mu_1}|x_1'\rangle \langle x_2|A_{\mu_2}|x_2'\rangle \right\rangle_{\mathrm{MM}}$$
$$= \int dk_1 dk_1' dk_2 dk_2' \exp(i(x_1k_1 - x_1'k_1' + x_2k_2 - x_2'k_2'))$$
$$\delta(k_1' - k_2)\delta(k_2' - k_1) \left\langle \tilde{A}_{\mu_1}(k_1 - k_1')\, \tilde{A}_{\mu_2}(k_2' - k_2) \right\rangle_{FT}$$
$$= \int dk_1 dk_2 \exp(i(x_1 - x_2')k_1 + (x_2 - x_1')k_2))$$
$$\left\langle \tilde{A}_{\mu_1}(k_1 - k_2)\, \tilde{A}_{\mu_2}(k_2 - k_1) \right\rangle_{FT} \qquad \leftarrow k_1 = q + \frac{p}{2}, k_2 = q - \frac{p}{2}$$
$$= \delta(\xi_1 + \xi_2) \int dp \exp(p \cdot (X_1 - X_2)) \left\langle \tilde{A}_{\mu_1}(p)\, \tilde{A}_{\mu_2}(-p) \right\rangle_{FT}$$

Here, $\xi_1 = x_1 - x_1'$, $X_1 = \frac{1}{2}(x_1 + x_1')$, similarly for ξ_2, X_2.



For L pt. Green's functions, we can show

$$\left\langle \langle x_1 | A_{\mu_1} | x_1' \rangle \cdots \langle x_L | A_{\mu_L} | x_L' \rangle \right\rangle_{\mathrm{MM}}$$
$$= \delta(\xi_1 + \cdots + \xi_L)\, G(X_1, \mu_1, \cdots, X_L, \mu_L)_{\mathrm{FT}}$$

This is true when the diagonal elements in the momentum representation are ignored and only planar diagrams are considered.

Instead of the color factor $\mathrm{tr}(t^{a_1} \cdots t^{a_L})$, we have $\delta(\xi_1 + \cdots + \xi_L)$.

If the diagonal elements in the momentum representation are ignored and only planar diagrams are considered, the matrix model is equivalent to large-N gauge theory.

N^2 DOF of adjoint rep. ⇔ continuous DOF of ξ

There is no negative norm state.

$\{|\xi\rangle ; \xi \in \mathbb{R}^{d-1,1}\}$ spans a positive definite Hilbert space for $\langle \xi | \xi' \rangle = \delta^d(\xi - \xi')$.

Gravitational DOF must arise from the dynamics of diagonal elements and non-planar diagrams.

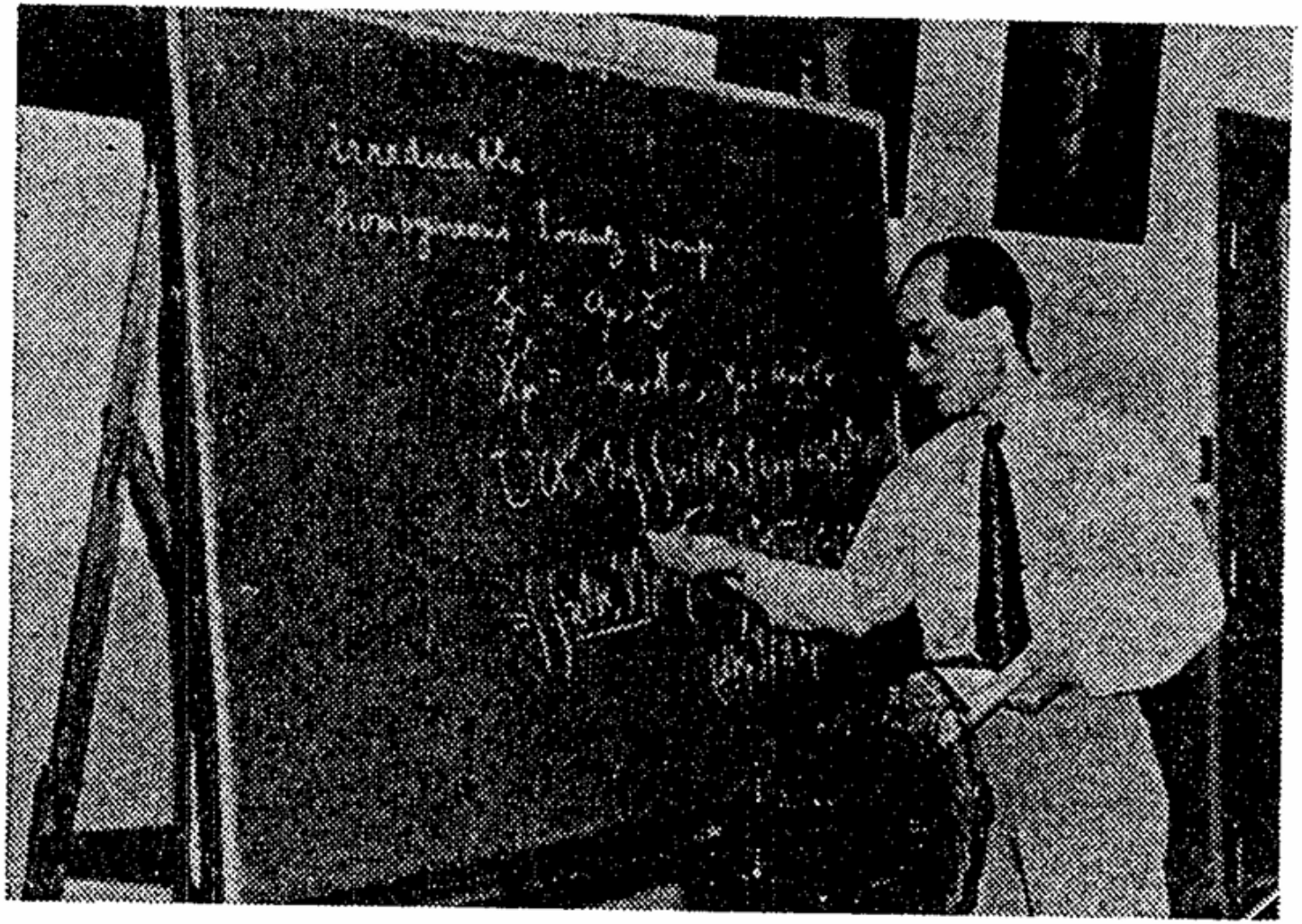
Yukawa and bi-local field

For some reason, Yukawa did not like renormalization theory and instead tried to understand hadrons by extended fundamental object, that is,nonlocal field theory.

Starting in 1950, he tried many kinds of nonlocal fields.
Bilocal field, Tetralocal field, .. , Elementary domain

Phys, Rev. 77 (1950) 219, 80 (1950) 1047, …

$$\left(\frac{\partial^2}{\partial X^{\mu 2}} - m^2\right)\psi(X,\xi) = 0,$$
$$(\xi^2 - \lambda^2)\psi(X,\xi) = 0,$$
$$\xi^\mu \frac{\partial}{\partial X^\mu}\psi(X,\xi) = 0.$$



京大における講演 (1950)

JPS journal, vol 37, no.4 (1982), p.276　By O. Hara

Yukawa seemed to try many possibilities of extended object except string theory.

The matrix model in coordinate representation is nothing but a bilocal field.

One of the nice things about the approach from the matrix model is that it has manifest invariance under (volume-preserving) diffeomorphisms before introducing background.
Thanks to this, gravity is naturally included.

Graviton as Nambu-Goldstone boson

5. Graviton and NG boson

Sometimes it is said that gravitons can be regarded as NG bosons.

e.g. P. R. Phillips, "Is the Graviton a Goldstone Boson?," Phys.Rev. 146 (1966) 966.

The argument can be general, but for simplicity let's consider the Einstein-Hilbert action.

(1) The action is diffeomorphism invariant.

(2) Then the action for constant $g_{\mu\nu}$ is $GL(d)$ invariant.

$$g_{\mu\nu} \to g'_{\mu\nu} = M_\mu{}^{\mu'} M_\nu{}^{\nu'} g_{\mu'\nu'} \text{ , for } M_\mu^\nu \in GL(d).$$

(3) The vacuum $g_{\mu\nu} = \eta_{\mu\nu}$ breaks $GL(d)$ to $SO(d-1,1)$.

(4) Therefore we have NG boson corresponding to

$GL(d)/SO(d-1,1) \cong$ rank 2 symmetric tensor ,

which is nothing but $h_{\mu\nu}$, $g_{\mu\nu} = \eta_{\mu\nu} + h_{\mu\nu}$.

The rest is as usual.

(5) Due to the diffeomorphism invariance, DOF of a vector field can be eliminated. For example, h_{0i} can be gauged away, and we have only h_{ij} .

(6) After taking the constraint from gauge fixing into account, we have the ordinary gravitons.

6. VPD in MM and GR

To be concrete we formally express the bilocal fields as differential operators of infinite degrees.

$$A_\mu = a_\mu^{(0)}(x) + a^{(1)}{}_\mu^{\nu}(x)\partial_\nu + a^{(2)}{}_\mu^{\nu_1\nu_2}(x)\partial_{\nu_1}\partial_{\nu_2} + \cdots.$$

(1) The action is invariant under VPD.

(2) Therefore the action for constant $a^{(n)}{}_\mu^{\nu_1\cdots\nu_n}(x)$ is invariant under global $SO(d-1,1)\otimes SL(d)$. Here μ transforms as $SO(d-1,1)$ vector while each ν_i as $SL(d)$ contravariant vector.

(3)The vacuum $a^{(1)}{}_\mu^{\nu}(x) = \eta_{\mu\nu},\ \text{others} = 0$ breaks $SO(d-1,1)\otimes SL(d)$ to diagonal $SO(d-1,1)$.

(4) Therefore we have NG bosons corresponding to

$SO(d-1,1) \otimes SL(d)$/diagonal $SO(d-1,1)$

$\cong SL(d) \cong$ rank 2 traceless tensor .

(5) This time the gauge symmetry is VPD, which is generated by a divergence free vector field $\partial_\mu v^\mu = 0$.

(6) The remaining DOF after gauge fixing is the same as the ordinary gravity plus rank 2 antisymmetric tensor.

(7) Therefore, GR +Kalb-Ramond fields appear as a consequence of VPD.

However, as we will see, $a^{(1)}{}_\mu^\nu(x)$ also has DOF of dilaton in addition to GR+KR.

A simplified model

7. Matrices as differential operators

We express the bi-local fields as a formal power series of differential operators.

$$A_\mu = a_\mu^{(0)}(x) + a^{(1)}{}_\mu^{\nu}(x)\partial_\nu + a^{(2)}{}_\mu^{\nu_1\nu_2}(x)\partial_{\nu_1}\partial_{\nu_2} + \cdots.$$

Then the EOM

$$\Big[A_\nu, \big[A_\mu, A_\nu\big]\Big] = 0,$$

becomes coupled differential equations for the local fields

$$a_\mu^{(0)}(x),\ a^{(1)}{}_\mu^{\nu}(x),\ a^{(2)}{}_\mu^{\nu_1\nu_2}(x), \cdots .$$

In general, these are complicated equations involving infinitely many fields.

Differential operators of degree 1

However, the set of differential operators of degree 1 forms a closed Lie algebra.

More explicitly, if A and B are of degree 1,

$$A = a^{(0)}(x) + a^{(1)^\nu}(x)\partial_\nu, \quad B = b^{(0)}(x) + b^{(1)^\nu}(x)\partial_\nu \ ,$$

their commutator $[A, B]$ is also of degree 1,

$$[A, B] = c^{(0)}(x) + c^{(1)^\nu}(x)\partial_\nu,$$

where $c^{(0)} = a^{(1)^\nu}\partial_\nu b^{(0)} - b^{(1)^\nu}\partial_\nu a^{(0)},$

$$c^{(1)^\mu} = a^{(1)^\nu}\partial_\nu b^{(1)^\mu} - b^{(1)^\nu}\partial_\nu a^{(1)^\mu} \ .$$

Therefore, the EOM $\left[A_\nu, \left[A_\mu, A_\nu\right]\right] = 0$ gives closed coupled equations.

Lie algebra of diffeomorphysms

We can restrict the operator to differential operator of degree 1 with no degree 0 terms. This is nothing but the Lie algebra of diffeomorphisms.
More concretely, if

$$A = a^{\nu}(x)\partial_{\nu}, \quad B = b^{\nu}(x)\partial_{\nu} \ ,$$

their commutator $[A, B]$ is also of this form,

$$[A, B] = c^{\nu}(x)\partial_{\nu},$$

where $c^{\mu} = a^{\nu}\partial_{\nu}b^{\mu} - b^{\nu}\partial_{\nu}a^{\mu} =: [a, b]^{\mu}.$

Lie algebra of VPD

We can further restrict the operator to the Lie algebra of VPD.

$$\text{Lie algebra of VPD} = \left\{ v^\mu \partial_\mu,\ \partial_\mu v^\mu = 0. \right\}$$

In fact we can easily check that if

$$A = a^\nu(x)\partial_\nu,\ \ \partial_\mu a^\mu = 0,\ \ B = b^\nu(x)\partial_\nu,\ \ \partial_\mu b^\mu = 0\ ,$$

their commutator $[A, B] = [a, b]^\mu \partial_\mu$

also satisfies $\partial_\mu [a, b]^\mu = \partial_\mu (a^\nu \partial_\nu b^\mu - b^\nu \partial_\nu a^\mu) =: 0.$

This is nothing but the fact that the set of VPD's forms a group.

8. Minimal model

As we have seen, the IIB matrix model in the coordinate representation has VPD invariance.
Therefore, one of the most simplified models is a model in which we assume all the matrices have the form of VPD.

$$A_\mu = a_\mu{}^\nu(x)\partial_\nu, \quad \partial_\nu a_\mu{}^\nu(x) = 0.$$

We call it the minimal model.

EOM $\left[A_\nu, \left[A_\mu, A_\nu\right]\right] = 0$

Gauge transformation

$$\delta_v A_\mu = \left[v, A_\mu\right], \; v = v^\mu \partial_\mu \, , \partial_\mu v^\mu = 0$$

Fate of the higher spin fields

In general, we can introduce constant background to the higher spin fields.

$$A_\mu = a_\mu^{(0)}(x) + a^{(1)}{}_\mu^{\nu}(x)\partial_\nu + a^{(2)}{}_\mu^{\nu_1\nu_2}(x)\partial_{\nu_1}\partial_{\nu_2} + \cdots.$$

In terms of the momentum representation, such background is expressed as

$$\langle k|A_\mu|k'\rangle = \left(c_\mu^{(0)} + c^{(1)}{}_\mu^{\nu}k_\nu + c^{(2)}{}_\mu^{\nu_1\nu_2}k_{\nu_1}k_{\nu_2} + \cdots\right)\delta(k-k') = F_\mu(k)\delta(k-k')$$

In the original matrix representation, this is nothing but to consider a general distribution of the diagonal elements.

matrix representation

$$\left\langle i\left|A_\mu^{(0)}\right|j\right\rangle = \begin{cases} p_\mu^i\,\delta_{ij} & (\mu = 1\sim d) \\ 0 & (\mu = d+1\sim 10) \end{cases}, \qquad \left\langle i\left|\psi^{(0)}\right|j\right\rangle = 0$$

momentum representation

$$\left\langle k\left|A_\mu^{(0)}\right|k'\right\rangle = \begin{cases} k_\mu\delta(k-k') & (\mu = 1\sim d) \\ 0 & (\mu = d+1\sim 10) \end{cases}, \quad \left\langle k\left|\psi^{(0)}\right|k'\right\rangle = 0.$$

For the IIB matrix model the effective action for the diagonal elements of A_μ and ψ is rather complicated even in the one-loop level. It is possible that a uniform distribution of p_μ^i $(i = 1,2,\dots)$ gives the minimum of the effective action.

Aoki-Iso-HK-Kitazawa-Tada

Kim-Nishimura-Tsuchiya

Ambjorn-Anagnostopoulos-Bietenholz-Hotta-Nishimura

If it is the case, the back ground for higher spin fields are all $\mathbf{0}$.

$$\langle k|A_\mu|k'\rangle A_\mu = \left(c_\mu^{(0)} + c^{(1)}{}_\mu^{\ \nu} k_\mu + c^{(2)}{}_\mu^{\ \nu_1\nu_2} k_\mu k_\nu + \cdots\right)\delta(k-k')$$

In other words, all the higher spin fields are massive.

(Note that $c_\mu^{(0)} = 0$ from the Lorentz invariance.)

Therefore it is natural to consider the minimal model as a low energy effective theory.

Unitarity of the minimal model

9. Weak field expansion

In order to check the unitarity of the minimal model, let's consider weak field expansion

$$A_\mu = a_\mu{}^\nu(x)\partial_\nu, \quad a_\mu{}^\nu = \delta_\mu{}^\nu + c_\mu{}^\nu \ , \ \partial_\nu c_\mu{}^\nu = 0 \ .$$

Then the EOM $\left[A_\nu, \left[A_\mu, A_\nu\right]\right] = 0$ becomes

$$\begin{aligned}\left[A_\mu, \ A_\nu\right]^\rho &= \left(\delta_\mu{}^\lambda + c_\mu{}^\lambda\right)\partial_\lambda\left(\delta_\nu{}^\rho + c_\nu{}^\rho\right) - (\mu \leftrightarrow \nu) \\ &= \partial_\mu c_\nu{}^\rho - \partial_\nu c_\mu{}^\rho + O(c^2)\end{aligned}$$

$$\left[A_\nu, \left[A_\mu, A_\nu\right]\right]^\lambda = \partial_\nu\left(\partial_\mu c_\nu{}^\lambda - \partial_\nu c_\mu{}^\lambda\right) + \mathrm{O}(\mathrm{c}^2) = 0$$

The gauge transformation becomes

$$\begin{aligned}\delta_v \, c_\mu{}^\lambda = \left[v, \ A_\mu\right]^\lambda &= v^\nu\partial_\nu\left(\delta_\mu{}^\lambda + c_\mu{}^\lambda\right) - \left(\delta_\mu{}^\nu + c_\mu{}^\nu\right)\partial_\nu v^\lambda \\ &= -\partial_\mu v^\lambda + O(c)\end{aligned}$$

EOM $\partial_\nu\left(\partial_\mu c_\nu^{\ \lambda} - \partial_\nu c_\mu^{\ \lambda}\right) = 0\,,\ \partial_\nu c_\mu^{\ \nu} = 0\,.$

Gauge transformation $\delta_v\, c_\mu^{\ \lambda} = -\partial_\mu v^\lambda\,, \partial_\mu v^\mu = 0.$

Using the gauge symmetry, we can fix the gauge to Lorenz like gauge: $\partial_\nu c_\nu^{\ \lambda} = 0.$

Then we have

$$\Box c_\mu^{\ \nu} = 0\,,\ \partial_\nu c_\mu^{\ \nu} = 0,\ \partial_\mu c_\mu^{\ \nu} = 0\,.$$

Thus we have massless states corresponding to a rank 2 tensor in $D-2$ dimensions, that is, graviton, K-R, and dilaton. This is the same as closed string.

Difficulty of the action

9. UV divergence of the action

EOM of the minimal model is very simple, and seems consistent.

However, there is a problem in writing down the action.

In fact for $A_\mu = a_\mu{}^\nu(x)\partial_\nu$ the action $\mathrm{S} = \mathrm{tr}\left(\left[A_\mu, A_\nu\right]^2\right)$ is not well defined.

This is because

$$\mathrm{tr}\left(u^\mu \partial_\mu \, v^\nu \partial_\nu\right)$$
$$= \int d^d x d^d y \, u^\mu(x)\partial_\mu \delta(x-y) \, v^\nu(y)\partial_\nu \delta(y-x)$$
$$= \infty \, .$$

The original action is finite if the bi-local fields $A_\mu(x,x')$ are smooth functions of x and x'.

$$S = -\int d^d x d^d y d^d z d^d w \{A_\mu(x,y)A_\nu(y,z)A_\mu(z,w)A_\nu(w,x) - A_\mu(x,y)A_\nu(y,z)A_\nu(z,w)A_\mu(w,x)\}$$

The problem arises from the delta function.

$$A_\mu(x,x') = a_\mu{}^\nu(x)\partial_\nu\delta(x-x')$$

More generally, the derivative expansion

$$A_\mu = a_\mu^{(0)}(x) + a^{(1)}{}_\mu^\nu(x)\partial_\nu + a^{(2)}{}_\mu^{\nu_1\nu_2}(x)\partial_{\nu_1}\partial_{\nu_2} + \cdots.$$

is useful for analyzing the EOM, but is problematic when considering the action.

10. UV regularization

The simple idea of regularizing trace with some UV cutoff does not work.

For example, if we regularize the trace as

$$\mathrm{tr}_{\mathrm{reg}}(A) = \mathrm{tr}(A \exp(\varepsilon \Box)),$$

the permutation invariance is violated

$$\mathrm{tr}_{\mathrm{reg}}(AB) \neq \mathrm{tr}_{\mathrm{reg}}(BA).$$

Then the EOM is no longer $\left[A_\nu, \left[A_\mu, A_\nu\right]\right] = 0.$

Another idea is to regularize operators themselves such as

$$(A)_{\mathrm{reg}} = A \exp\left(\frac{\varepsilon}{2}\Box\right).$$

Then the permutation invariance is satisfied, and the EOM is.

$$\left[(A_\nu)_{\mathrm{reg}}, \left[(A_\mu)_{\mathrm{reg}}, (A_\nu)_{\mathrm{reg}}\right]\right] = 0.$$

One problem with this is that the commutators of two operators contains higher derivatives.

$$\left[\left(a_\mu{}^\lambda \partial_\lambda\right)_{\mathrm{reg}}, \left(a_\nu{}^\rho \partial_\rho\right)_{\mathrm{reg}}\right] = \left(b_{\mu\nu}^\lambda \partial_\lambda + \varepsilon c_{\mu\nu}^{\alpha\beta\gamma} \partial_\alpha \partial_\beta \partial_\gamma\right) \exp(\varepsilon\Box).$$

Here,

$$b_{\mu\nu}^\lambda = a_\mu{}^\rho \partial_\rho a_\nu{}^\lambda - (\mu \leftrightarrow \nu), \quad c_{\mu\nu}^{\alpha\beta\gamma} = a_\mu{}^\alpha \partial^\beta a_\nu{}^\gamma - (\mu \leftrightarrow \nu).$$

The second term vanishes in the naïve $\varepsilon \to 0$ limit.

$$\left[\left(a_\mu{}^\lambda \partial_\lambda\right)_{\mathrm{reg}},\ \left(a_\nu{}^\rho \partial_\rho\right)_{\mathrm{reg}}\right] = \left(b_{\mu\nu}^\lambda \partial_\lambda + \varepsilon c_{\mu\nu}^{\alpha\beta\gamma} \partial_\alpha \partial_\beta \partial_\gamma\right) \exp(\varepsilon \square).$$

But it contributes in the same order as the first term after taking trace. This is because

$$\mathrm{tr}(\partial^{2n} \exp(\varepsilon \square)) \sim \varepsilon^{-n}\, \mathrm{tr}(\exp(\varepsilon \square))\ .$$

We again fail to derive the action that gives the EOM $\left[A_\nu, \left[A_\mu, A_\nu\right]\right] = 0.$

The situation could be described as follows.

If we express the YM type matrix model in terms of bilocal field, both the action and EOM are finite.

Expanding the bilocal field as a formal power series of derivatives, the EOM becomes a series of differential equations for higher spin fields.
This EOM has no divergent coefficient, but the action is no longer a finite functional of the fields.

The truncation of the formal power series at degree 1 gives a self consistent EOM for the spin 1 fields.
This is consistent in that it describes special solutions of the full system. However it is difficult to find a closed form of the action for the spin 1 fields.

However, this situation is common in loop equations (or more generally in the Schwinger-Dyson equation), as noted by Jevicki and Sakita.

Probably we can find a non-local action that replicates the EOM.

Summary

If the emergent spacetime can be described by matrices that fluctuate around diagonal matrices, it is useful to represent the matrices as bi-local fields.

The action of the IIB matrix model in terms of bilocal field can be regarded as a kind of pre-geometric theory. At the pre-geometric level, that is, before considering the emergent background, the theory is invariant under the VPD.

GR arises as a low energy effective theory consisting of NG bosons around the emergent geometry.

Technically, bilocal fields are more tractable than higher spin fields and may help us better understand the dynamics of matrix models.

Thank you very much.

Appendix

Relation to the tele-parallel gravity

Torsion and teleparallel gravity

The field strength of the minimal model $[A_a, A_b]$ is nothing but the torsion of the tele-parallel connection (Weitzenböck connection), if we identify A_a as the covariant derivative.

$$A_a = e_a^{\nu}\partial_{\nu}\,,\ [A_a, A_b] = T_{ab}^{\ \lambda}\partial_{\lambda}$$

Naïvely, the Yang-Mills type action is given by

$$S = \mathrm{tr}\left([A_a, A_b]^2\right) = \mathrm{tr}\left((T_{ab}^{\ \lambda}(x)\partial_{\lambda})^2\right)$$

$$= \int d^d x\, T_{ab}^{\ \lambda}(x)\, T_{ab}^{\ \rho}(x)\, \langle x|\partial_{\lambda}\partial_{\rho}|x\rangle$$

As we have seen any simple regularization does not satisfy the permutation invariance and fails to obtain the YM like EOM.

Here let's see what happens if we take a naïve covariant regularization:

$$\langle x|\partial_\lambda \partial_\rho|x\rangle = \Lambda^2 \sqrt{g} g_{\mu\nu},\ g^{\mu\nu} = e_a^\mu\, e_b^\nu\, \delta^{ab}. \leftarrow A_a = e_a^\nu \partial_\nu$$

Then the action becomes

$$S_1 = \Lambda^2 \int d^d x \sqrt{g} g_{\mu\nu}\, T_{ab}{}^{\mu}\, T_{ab}{}^{\nu}\ .$$

Because $T_{ab}{}^{\nu}$ is a rank 3 tensor, we can make two more actions that are invariant under diffeomorphisms.

They are

$$S_2 = \Lambda^2 \int d^d x \sqrt{g}\, e_\mu^{\ c} e_\nu^{\ b}\, T_{ab}^{\ \ \mu}\, T_{ac}^{\ \ \nu}\,,$$

$$S_3 = \Lambda^2 \int d^d x \sqrt{g}\, e_\mu^{\ b} e_\nu^{\ c}\, T_{ab}^{\ \ \mu}\, T_{ac}^{\ \ \nu}\,,$$

It can be shown that only the special combination

$$S_G = \frac{1}{4} S_1 + \frac{1}{2} S_2 - S_3$$

has invariance under local Lorentz transformation, and the action is equivalent to the Einstein-Hilbert action.

In other words, S_1 is not local Lorentz invariant and contains more DOF than gravity.

In the weak field limit, they look like Kalb-Ramond and dilaton fields. But the decoupling of the negative norm states from the S-matrix is not clear.

IIB Matrix Model

1. Nambu-Goto action and Schild action

The Nambu-Goto action

$$S_{\mathrm{NG}} = -\rho \int d^2\xi \, \sqrt{g} \sqrt{-\{X^\mu, X^\nu\}^2}$$

is equivalent to the Schild action

$$S_{\mathrm{Shild}} = \alpha \int d^2\xi \, \sqrt{g} \, \{X^\mu, X^\nu\}^2 \quad .$$

Here $\sqrt{g}$: volume density on the world sheet,

$\{X, Y\} = \frac{1}{\sqrt{g}} \varepsilon^{ab} \partial_a X \, \partial_b Y$: Poisson bracket.

Note: $\sqrt{g}$ is not dynamical. We can take any $\sqrt{g}$.
$\sqrt{g}$ is cancelled in S_{NG}.

String tension from Schild action

$$S_{\mathrm{NG}} = -\rho \int d^2\xi \, \sqrt{g}\sqrt{-\{X^\mu, X^\nu\}^2}$$

$$S_{\mathrm{Shild}} = \alpha \int d^2\xi \, \sqrt{g}\,\{X^\mu, X^\nu\}^2$$

EOM for the Schild action:

$$\delta S_{\mathrm{schild}} = 0 \;\Rightarrow \{X^\nu, \{X^\mu, X^\nu\}\} = 0\,.$$

From this, we can show

$$\partial_a \{X^\mu, X^\nu\}^2 = 0.$$

That is, $-\{X^\mu, X^\nu\}^2 =: C^2$ is a constant on the world sheet.

By comparing the actions we have

$$\rho = 2\alpha C = 2\alpha\sqrt{-\{X^\mu, X^\nu\}^2}.$$

ρ is determined by the initial values of X^μ.

Proof of $\{X^\mu, X^\nu\}^2$=const. for Schild action

From EOM $\{X^\nu, \{X^\mu, X^\nu\}\} = 0$, we have

$$\varepsilon^{ab}\partial_a X^\nu \partial_b \sigma^{\mu\nu} = 0, \quad \sigma^{\mu\nu} = \{X^\mu, X^\nu\}.$$

Multiplying $\partial_c X^\mu$, we have

$$\varepsilon^{ab}\,\partial_c X^\mu\,\partial_a X^\nu \partial_b \sigma^{\mu\nu} = 0.$$

Because $\sigma^{\mu\nu} = -\sigma^{\nu\mu}$, $\partial_c X^\mu\,\partial_a X^\nu \partial_b \sigma^{\mu\nu}$ is antisymmetric in a and c. Therefore we can replace

$$\partial_c X^\mu\,\partial_a X^\nu \;\text{ to }\; \varepsilon_{ca}\varepsilon^{ef}\partial_e X^\mu\,\partial_f X^\nu = \varepsilon_{ca}\sqrt{g}\,\sigma^{\mu\nu}$$

Thus we have

$$\sigma^{\mu\nu}\partial_c \sigma^{\mu\nu} = 0 \Rightarrow \partial_c(\sigma^{\mu\nu})^2 = 0\,.$$

Proof of equivalence of S_{NG} and S_{Schild}

We can rewrite S_{NG} as

$$S'_{\mathrm{NG}} = -\frac{\rho}{2}\int d^2\xi \left(\omega + \frac{1}{\omega}\left(-\Sigma^2\right)\right), \quad \Sigma^2 = \left(\varepsilon^{ab}\partial_a X^\mu \, \partial_b Y^\nu\right)^2 = \left(\sqrt{g}\sigma^{\mu\nu}\right)^2.$$

Here ω is a dynamical scalar density.

By the diffeomorphism invariance, we can choose a gauge such that $\omega = C\sqrt{g}$, where $\sqrt{g}$ is the fixed scalar density and C is a constant.

Then S'_{NG} is reduced to

$$S'_{\mathrm{NG}} = \alpha \int d^2\xi \sqrt{g}\, \sigma^2 , \quad \alpha = \frac{\rho}{2C} ,$$

with a constraint coming from the EOM for ω :

$$\Sigma^2 = -\omega^2 \Leftrightarrow \sigma^2 = -C^2.$$

Thus we have seen

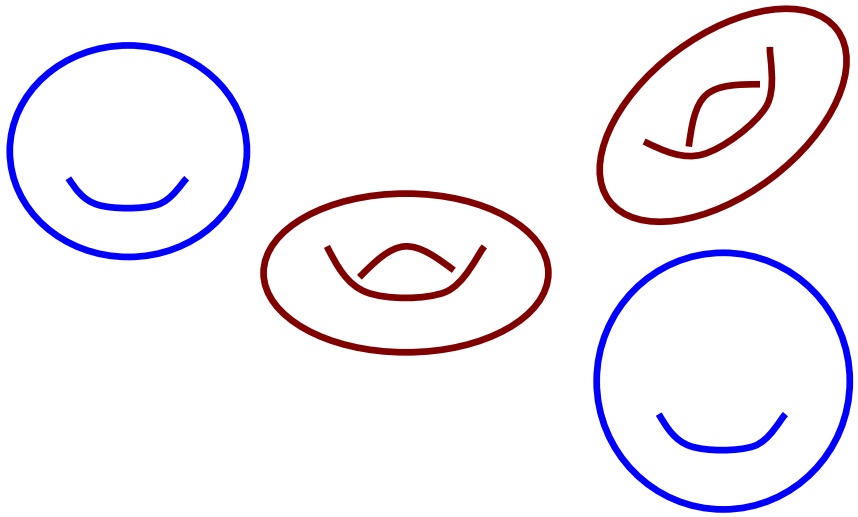
$$S_{\mathrm{NG}} = -\rho \int d^2\xi \sqrt{\Sigma^2} \;\Leftrightarrow\; \begin{cases} S_{\mathrm{Schild}} = \alpha \int d^2\xi \sqrt{g}\, \sigma^2, \;\; \alpha = \dfrac{\rho}{2C}, \\ \sigma^2 = -C^2. \end{cases}$$

But $\sigma^2 = \mathrm{const}$ automatically follows from the Schild action.

The Schild action contains the NG action with various string tensions.

The string tension is determined by the initial conditions.

String tension is a constant on each connected component of the world sheet.



But in the real universe, our worldsheet must be one big connected thing. This is because disconnected worldsheets never communicate. (Wheeler?)

2. Matrix regularization

The entire universe is described by one large world sheet with a two-dimensional phase space structure.

Then we introduce the matrix regularization.

We expect that the whole universe can be described by the Yang-Mills type matrix model:

$$S_{\text{matrix}} = -\alpha \frac{1}{4} Tr\left(\left[A_\mu, A_\nu\right]^2\right).$$

3. Type IIB string

We consider the type IIB string.

Green-Schwarz action in the NG form

$$S_{GS} = -\rho \int d^2\xi \Big(\sqrt{-\frac{1}{2}\Sigma^2} + i\varepsilon^{ab}\partial_a X^\mu \Big(\bar{\theta}^1 \Gamma_\mu \partial_b \theta^1 + \bar{\theta}^2 \Gamma_\mu \partial_b \theta^2 \Big) + \varepsilon^{ab} \bar{\theta}^1 \Gamma^\mu \partial_a \theta^1 \, \bar{\theta}^2 \Gamma_\mu \partial_b \theta^2 \Big),$$

X^μ $(\mu = 0 \sim 9)$

θ^1, θ^2 : 10D Mayorana-Weyl

$$\Sigma^{\mu\nu} = \varepsilon^{ab} \Pi_a^\mu \Pi_b^\nu,$$

$$\Pi_a^\mu = \partial_a X^\mu - i\bar{\theta}^1 \Gamma^\mu \partial_a \theta^1 + i\bar{\theta}^2 \Gamma^\mu \partial_a \theta^2$$

Gauge fixing for the κ-symmetry $\theta^1 = \theta^2 = \frac{1}{2}\psi$

$$S_{\mathrm{GS}} = -\rho \int d^2\xi \left(\sqrt{-\frac{1}{2}\Sigma^2} + \frac{i}{2}\bar{\psi}\varepsilon^{ab}\partial_a X^\mu \bar{\psi}\Gamma_\mu \partial_b \psi \right),$$

$$\Sigma^{\mu\nu} = \varepsilon^{ab}\partial_a X^\mu \partial_b X^\nu .$$

Schild action for type IIB string

$$S_{\mathrm{Shild}} = \alpha \int d^2\xi \sqrt{g} \left(\frac{1}{4} \{X^\mu, X^\nu\}^2 + \frac{i}{2} \bar{\psi}\Gamma^\mu \{X^\mu, \psi\} \right).$$

Matrix regularization

Applying the matrix regularization, we have

$$S_{\mathrm{IIB}} = \alpha\, Tr\left(-\frac{1}{4}\left[A_\mu, A_\nu\right]^2 - \frac{1}{2}\bar{\psi}\Gamma^\mu\left[A_\mu, \psi\right]\right).$$

IIB matrix model Ishibashi-HK-Kitazawa-Tsuchiya

The N=2 SUSY is maintained after this regularization.