Higher Representation Theory for Excitations in Gauge Theories

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Introduction

Today's message

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- Representation theory for **excitations**
  - **Higher groupoid representation**
  - Depends on **microscopic symmetry** and **infrared phase**
  - **Systematic** classification of excitations
  - **Higher** representation $\Rightarrow$ **extended** excitations

- Meaning of "gauge charge"
- ("Non-invertible" symmetries $\Rightarrow$ higher strip algebra representation)

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Representation theory in 1+1d (Cordova, Holfester, and Ohmori 2024)

Clay Córdova

Nicholas Holfester

Symmetry Preserved Phase

- G: finite group microscopic symmetry
- A single vacuum $|0\rangle$.
- Symmetry preserves the vacuum $g|0\rangle = |0\rangle$.
- Excitations are (and thus whole Hilbert space is)in a representation:

$$g \cdot |e\rangle = \rho_e(g)|e\rangle,$$

where $\rho_e(g)$ is a representation matrix.

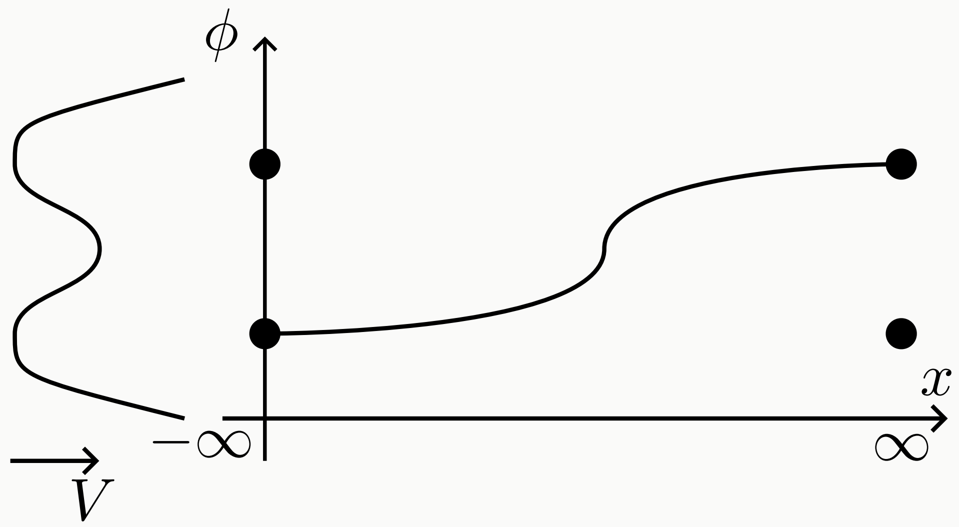
- Excitation $|e\rangle$ is classified by its **irreducible representation** of G.

Completely Broken Phase

- G: finite group (e.g. $\mathbb{Z}_2, S_3, \cdots$)
- Complete spontaneous symmetry breaking: $G \rightsquigarrow \{*\}$.
- $|G|$ degenerate vacua.
- $v \in G \Rightarrow$ vacuum $|v\rangle$.
  - Microscopic symmetry $g \in G$ acts as $g|v\rangle = |gv\rangle$.

Solitons in SSB phase

- Soliton $|s(v_1, v_2)\rangle$ interpolates between v_1 and v_2.
- $g|s(v_1, v_2)\rangle = |s(gv_1, gv_2)\rangle$.
- G-action generates **$|G|$-fold multiplet**.



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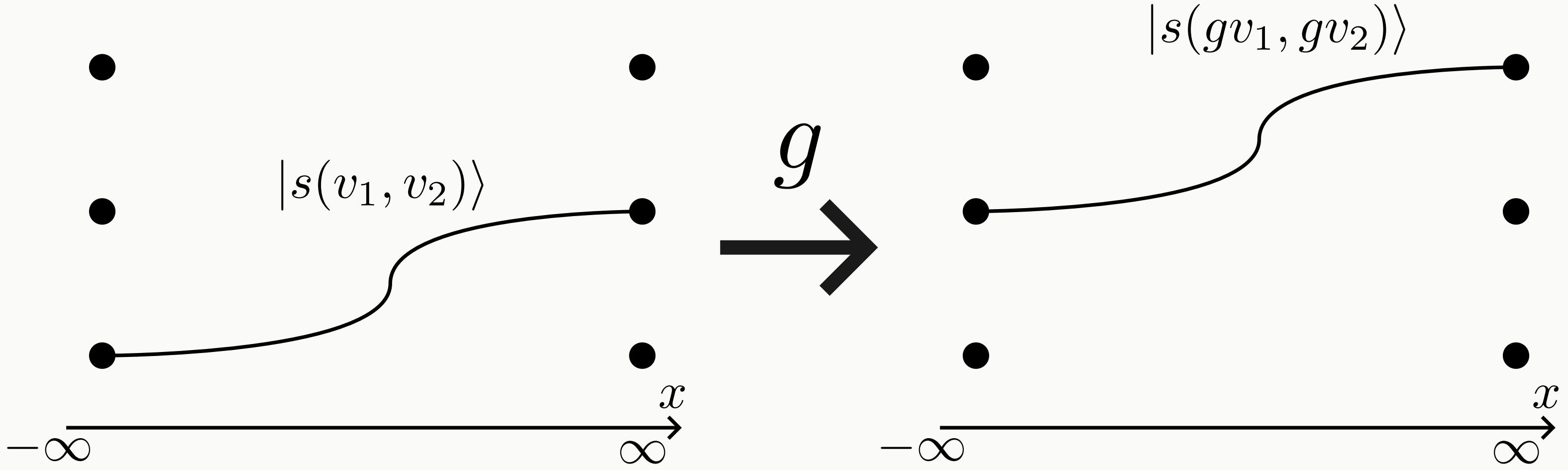

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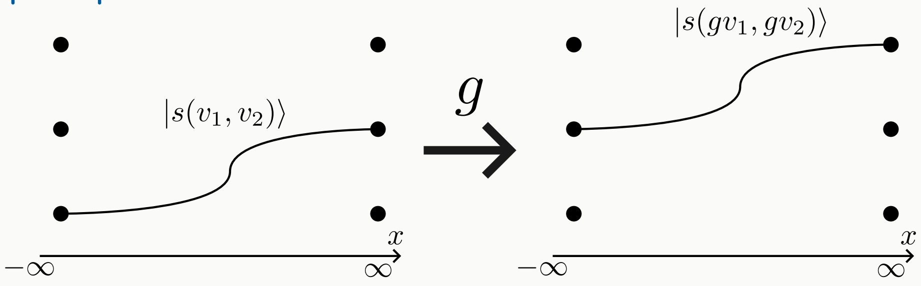


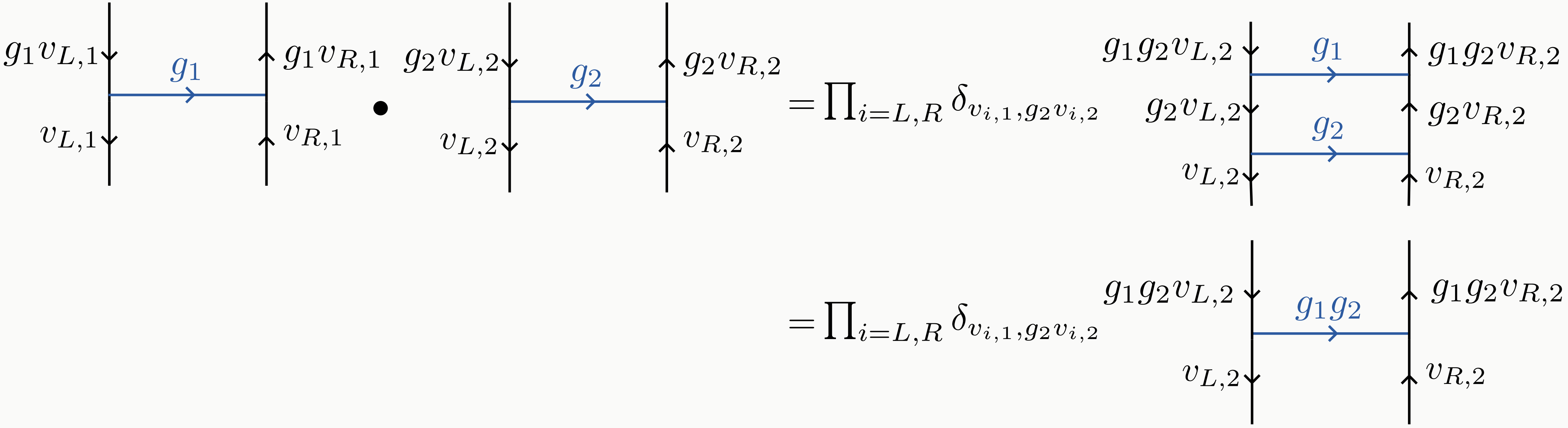
Symmetry is "non-linearly represented".
A better understanding is that states in the phase are in (linear) representations of a **groupoid algebra**.

Groupoid Algebra

- The groupoid algebra $\mathbb{C}[X /\!/ G]$: a vector space spanned by $\xleftarrow{g} v$:
  - $v \in X$ is a vacua (at the left ∞, say).
  - i.e. $\mathbb{C}[X /\!/ G] \ni \sum_{g,v} c_{g,v}[\xleftarrow{g} v]$
  - Multiplication: $[\xleftarrow{g_1} v_1] \cdot [\xleftarrow{g_2} v_2] = \delta_{v_1, g_2 v_2}[\xleftarrow{g_1 g_2} v_2]$.
- **Representation of groupoid algebra** (R, ρ) is an algebra hom $\rho : \mathbb{C}[X /\!/ G] \to \mathrm{GL}(R)$ for a Hilbert space R.
- For $X = *$, (R, ρ) is a usual group representation.

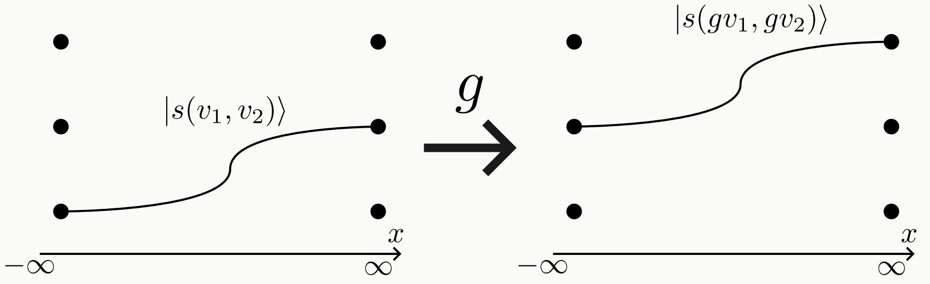
Groupoid for SSB

- For the SSB phase where G is completely broken, the relevant groupoid is $G^2 /\!\!/ G$, where the action is $(v_1, v_2)g = (v_1 g, g^{-1} v_2)$.
- $(v_1, v_2) \in G^2$: vacua at left and right infinity.
- Multiplication in $\mathbb{C}[G^2 /\!\!/ G]$:

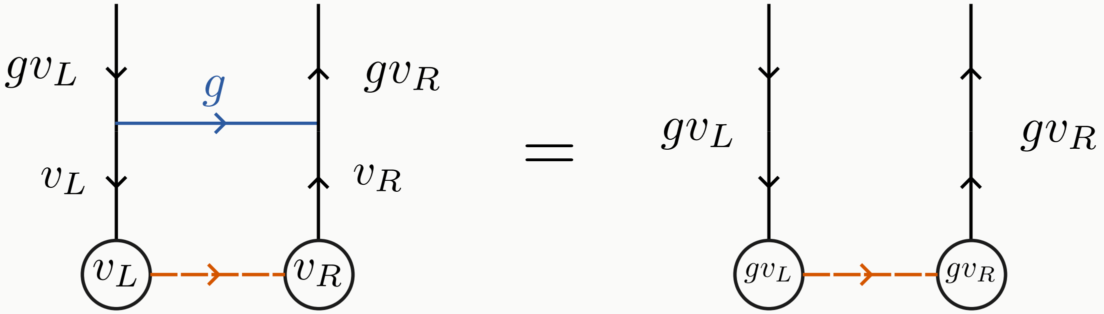


Groupoid Algebra Representation

- $[\overset{g}{\leftarrow}(v_L, v_R)]|s(v'_L, v'_R)\rangle = \delta_{v_L,v'_L}\delta_{v_R,v'_R}|s(gv_L, gv_R)\rangle$

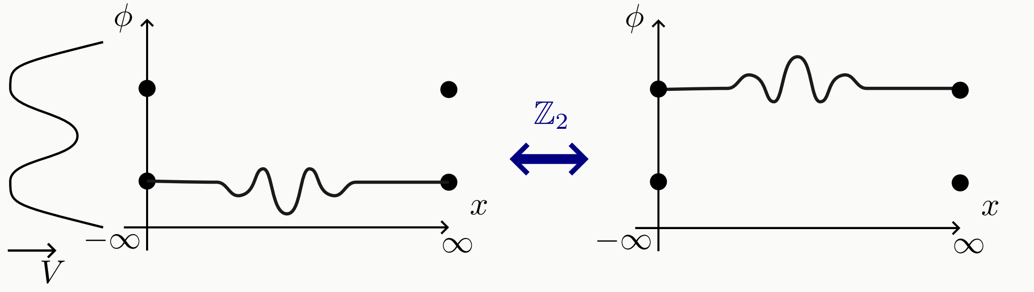


- Diagram:

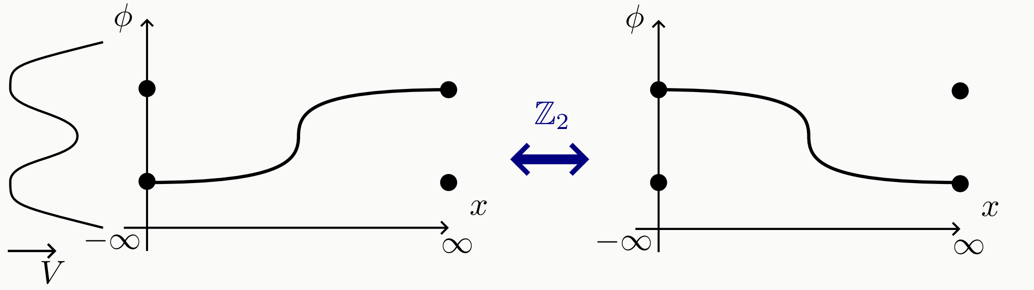


Groupoid Algebra Representation for $\mathbb{Z}_2$ SSB

- Irreps of $\mathbb{C}[\mathbb{Z}_2^2 /\!\!/ \mathbb{Z}_2]$: R_0, R_1.
  - R_0 : "particle" sector : $(v_L, v_R) = (0,0), (1,1)$



  - R_1 : "soliton/antisoliton" sector : $(v_L, v_R) = (0,1), (1,0)$.

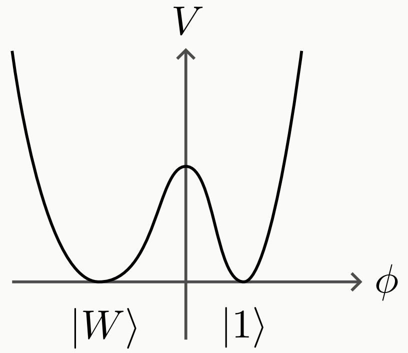


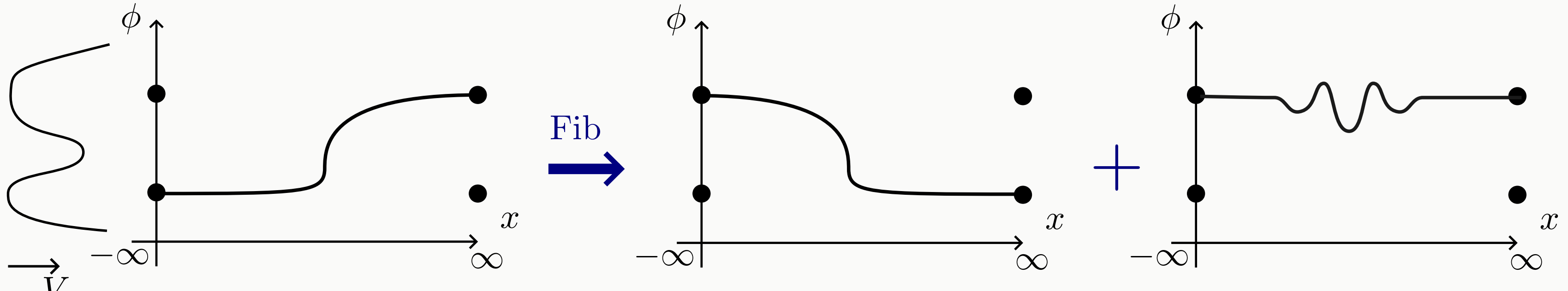
Rep. Theory for Solitons w/ Non-inv. Sym.

(Cordova, Holfester, and Ohmori 2024)

- Generalizes to **non-invertible symmetry**
- $\rightsquigarrow$ "**strip algebra**" (Kitaev and Kong 2012; Johnson-Freyd and Reutter WIP)
- E.g. Deformed tricritical Ising model (Zamolodchikov 1990)
  - SSB'en "Fibonacci" sym. (Chang et al. 2019)
  - a multiplet involving **both** particles and solitons!

  (Córdova, García-Sepúlveda, and Holfester 2024)





General Theory

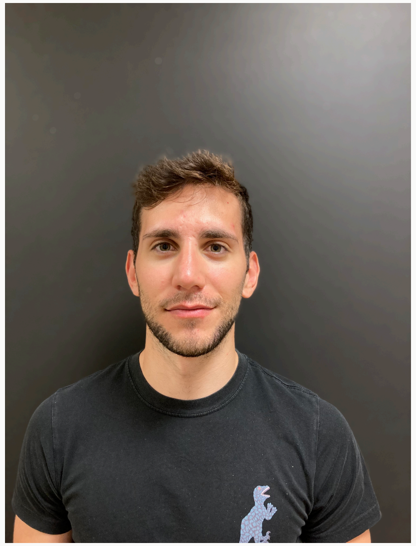
symbol	**math**	**physics**
$\mathcal{C}$	fusion category	UV symmetry/ Topological lines
$\mathcal{M}$	module category	IR gapped phase
$\mathbf{Str}_{\mathcal{C}}(\mathcal{M})$	weak Hopf alg.	Symmetry action on $\mathbb{R}^2$
$\mathrm{Rep}_{\mathcal{C}}(\mathcal{M}) \cong \mathrm{End}_{\mathrm{Mod}(\mathcal{C})}(\mathcal{M})$	Dual Category	**Excitation multiplets**

While $\mathbf{Str}_{\mathcal{C}}(\mathcal{M})$ is often cumbersome, one can often directly compute the representation category $\mathrm{Rep}_{\mathcal{M}}(\mathcal{C})$. E.g. full SSB: $\mathrm{Rep}_{\mathcal{C}}(\mathcal{C}) \cong \mathcal{C}$.

Representation Theory for Gauge Theories

Based on (Gagliano, Grigoletto, and Ohmori 2025, WIP)

Finn Gagliano



Andrea Grigoletto

Higher Form Symmetry in Gauge theory

- pure YM with non-ab. gauge group $G \Rightarrow Z(G)^{(1)}$ **form symmetry**
  - e.g. $G_{\text{gauge}} = SU(N) : \mathbb{Z}_N^{(1)}$ one-form symmetry
  - $W_\square$ cannot be screened by gauge fields.

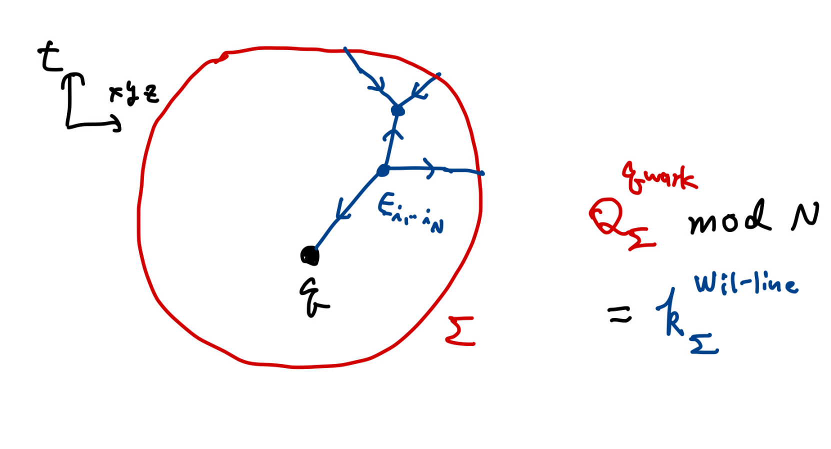
> **confinment** (or not) $\Leftrightarrow \mathbb{Z}_N^{(1)}$ is preserved (or SSB′en).

Symmetry in single flavor QCD

- $G_{\text{gauge}} = \mathrm{SU}(N)$ + a (scalar) quark q in $\square$
- $U(1)_q : q \mapsto e^{i\theta} q$: quark rotation
- $(\mathbb{Z}_N)_q = Z(G_{\text{gauge}}) \subset G_{\text{gauge}}$: gauged!
- $G_{\text{UV}} = U(1)_B = U(1)_q/\mathbb{Z}_N$: **Baryon number**
  - A local (gauge invariant) operator has **integer** Baryon charge.
  - $q_{i_1} \partial q_{i_2} \cdots \partial^N q_{i_N} \epsilon^{i_1,\cdots,i_N}$

Symmetry Matching for Heavy Quark

- m_q: quark mass
  - $\mu \leq m_q$: $\mathbb{Z}_N^{(1)}$: **emergent** one-form symmetry
- Symmetry matching homomorphism: $U(1)_B \xrightarrow{\beta_N} \mathbb{Z}_N^{(1)}$
  - **Non-trivial!** (Bockstein map) c.f. (Seiberg and Seifnashri 2025)



Adjoint Higgs Phase

- $G_{\mathrm{UV}} = U(1)_B \xrightarrow{\mu < m_q} \mathbb{Z}_N^{(1)} \xrightarrow{\mu < v_\Phi} *$
- "Exact sequence": $U(1)_q \to U(1)_B \to \mathbb{Z}_N^{(1)}$
  - "$\mathbb{Z}_N^{(1)} = U(1)_B / U(1)_q$"
  - "break **up**" : $U(1)_B \rightsquigarrow U(1)_q$
- $\mathrm{Rep}_{U(1)_B}(\mathrm{Adj.\ Higgs}) \cong 3\mathrm{Rep}(U(1)_q) \boxtimes_\omega 3\mathrm{Vect}_{\mathbb{Z}_N^{(1)}}$
- **quark** : $\rho \in 3\mathrm{Rep}(H = U(1)_q)$
- **center vortex**: $\sigma \in 3\mathrm{Vect}(G/H = \mathbb{Z}_N^{(1)})$.

Summary

Message

Represntation theory for excitations **depends on the phase**

- Discret symmetery SSB phase $\Rightarrow$ groupoid algebra representation
- 1+1d SSB phase $\Rightarrow$ **solitons**
- (scalar) QCD
  - One-form symmetry pres / SSB
    $\Rightarrow$ **Representation theory** does not/ does contain quark

Prospects

> Our work: **bosonic** theory in **gapped** and **zero-temperature** phases.

- Generalizations
  - Thermal Phase
    - Full QCD phase?
  - Gapless phases
    - Full gauge charge in Coulomb phase?
    - Quantum Gravity?

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Our work: **bosonic** theory in **gapped** and **zero-temperature** phases.

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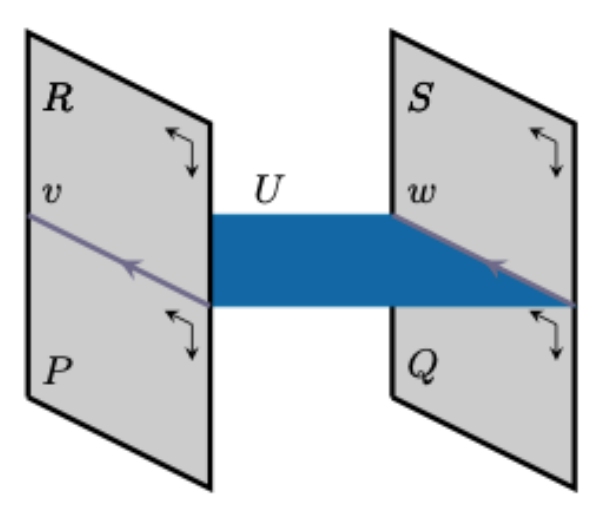
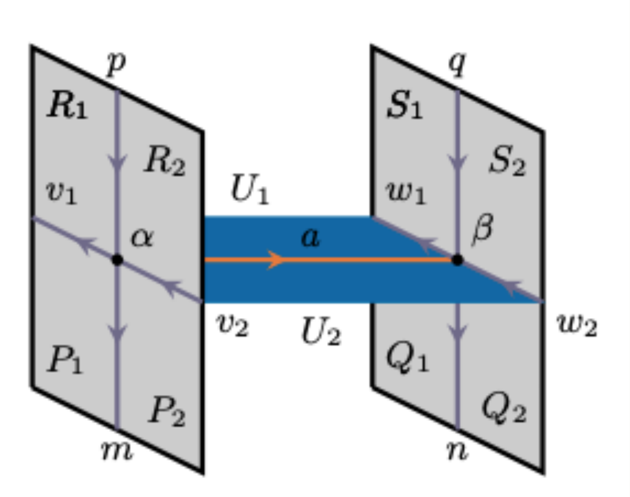
Thank you !!

Higher Strip Algebra

- In $d+1$ QFT, higher strip algebra:
  - an weak Hopf algebra object in $d\mathrm{Vect}$.
  - Rep. theory $\cong \mathrm{Vect}_{G/H} \otimes_\omega \mathrm{Rep}(H)$



object ($d=2$)



1-morphism ($d=2$)