Quantum and Holographic Aspects of Spacetime Anisotropy

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based on 2312.16284, 2409.06667, 2507.14996

Outline

I. Lifshitz Anisotropy (2)

II. Properties of Lifshitz Field Theory (LFT) (7)

III. Holography for Boundary Lifshitz Field Theory (AdS/BLFT) (5)

IV. Lifshitz Anisotropic Josephson Junction (6)

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Lifshitz anisotropy

- An interesting instance of anisotropy is when physics is invariant under the *Lifshitz scaling transformation*

$$t \to \lambda^z t, \quad x^i \to \lambda x^i, \quad \lambda > 0.$$

  Here spatial isotropy is maintained while time is anisotropic for an anisotropy exponent $z \neq 1$. E.g. critical point of RG

- Lifshitz scale invariant system can be recognized by the dispersion relation of its massless modes:

$$\omega \sim ck^z,$$

- Examples of Lifshitz invariant systems (all $z = 2$):
  - tricritical point of liquid helium (Lifshitz)
  - modified gravity scenario with ghost condensate (Arkani-Hamed 0312099).
  - critical point of the Rokhsar-Kivelson (RK) quantum dimer model: continuum $z = 2$ Lifshitz scalar field theory emerges

$$S = \frac{1}{2} \int dt d^2x \left[(\partial_t \phi)^2 - \kappa^2 (\nabla^2 \phi)^2 \right]$$

- The point $z = 2$ is somewhat special since the Lifshitz scaling transformation combines with the Galilean boost to form the Schrodinger group only for this value of z.

Q. What about $z \neq 2$?

We will look at general z both field theoretically and holographically.

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RK ground state

- Consider massless Lifshitz scalar field theory in $(1+1)$-dimensions

$$S = \frac{1}{2}\int dtdx \left[(\partial_t\phi)^2 - \kappa^2(\nabla_x^z\phi)^2\right], \quad \text{where } \nabla_x^z e^{ikx} := (ik)^z e^{ikx}.$$

S is invariant under the Lifshitz scaling transformation, plus

$$\phi \to \lambda^{-\Delta_\phi}\phi, \quad \text{where} \quad \Delta_\phi = \frac{d-z}{2}.$$

- Hamiltonian can be put into factorized form

$$H = \int dx A^\dagger(x)A(x) + E_{\text{vac}},$$

where

$$A(x) = \frac{1}{\sqrt{2}}\left(i\Pi(x) + \kappa\nabla_x^z\phi(x)\right),$$

are generalized annihilation and creation operators.

- A ground state of the theory may be defined by

$$A(x)|\Psi_0\rangle = 0, \quad \forall x.$$

- This can be solved explicitly

$$|\Psi_0\rangle = \frac{1}{\sqrt{\mathcal{Z}}} \int \mathcal{D}\phi \; e^{-S_{\rm cl}[\phi]/2} |\phi\rangle, \quad S_{\rm cl}[\phi] := \kappa \int \left(\nabla_x^{\frac{z}{2}} \phi\right)^2 dx,$$

  where $\mathcal{Z} = \int \mathcal{D}\phi e^{-S_{\rm cl}[\phi]}$ is a normalization factor and $S_{\rm cl}$ is the action for an auxillary QM system.
- This special kind of ground state known as the RK vacuum and has interesting properties.

Interesting properties of RK vac

(Basak-Chakraborty-Chu-Giataganas-Parihar 2312.16284)

1. 2d-1d correspondence: Expectation value of operators at equal time is given by the path integral of a 1d system

$$\langle \Psi_0 | O_1(\phi(x_1)) O_2(\phi(x_2)) \cdots | \Psi_0 \rangle = \frac{1}{\mathcal{Z}} \int \mathcal{D}\phi e^{-S_{\rm cl}[\phi]} O_1(\phi(x_1)) O_2(\phi(x_2)) \cdots .$$

2. Family of degenerate vacuum:
   - As usual, one can evaluate the functional integration with the help of the path integral propagator

$$K(\phi_i, \phi_f; x_i, x_f) = \int_{\phi(x_i)=\phi_i}^{\phi(x_f)=\phi_f} \mathcal{D}\phi \exp\left(-\kappa \int_{x_i}^{x_f} \left(\nabla_x^{\frac{z}{2}} \phi \right)^2 dx \right)$$

   where ϕ is a solution to the classical EOM

$$\nabla_x^z \phi = 0.$$

- This has the general solution

$$\phi(x) = \phi_i + (\phi_f - \phi_i)g(s), \quad s := (x - x_i)/l, \quad l = x_f - x_i,$$

with $g(s)$ given by the "polynomial"

$$g(s) := \sum_{n=1}^{[z]} c_n s^{z-n}, \quad \sum_n c_n = 1.$$

- Therefore, the evaluation of the path integral requires a specification of the polynomial g.

Meaning that the Lifshitz theory has a family of ground states whose degeneracy is parameterized by a function g.

Entanglement

Q. How is quantum entanglement affected by anisotropy?

- Even though there is no twisted operator, Reyni entanglement entropy for the RK vac $\rho = |\Psi_0\rangle\langle\Psi_0|$ can be computed using path integral and replica technique. We find

$$S(A) = \frac{z-1}{2}\log\frac{l}{\epsilon} + \frac{c_1}{2}, \quad S_n(A) = \frac{z-1}{2}\log\frac{l}{\epsilon} + \frac{c_n}{2}$$

- Note that

$$\mathrm{RKvacuum}: \quad [S_n(A)]_{\mathrm{UV}} = [S(A)]_{\mathrm{UV}}.$$

This is different from the usual case of a conformal vacuum:

$$\mathrm{CFTvacuum}: \quad [S_n(A)]_{\mathrm{UV}} = \frac{1}{2}(1+1/n)[S(A)]_{\mathrm{UV}}.$$

Next: use the field theory to learn something about the holography side.

Lifshitz Holography

- It has been proposed that 2d LFT is dual to the 3d gravity theory with the Lifshitz metric (Kachru 2008)

$$ds^2 = L^2\left[-\frac{dt^2}{r^{2z}} + \frac{dr^2}{r^2} + \frac{dx^2}{r^2}\right].$$

The isometry is consistent with the Lifshitz symmetry:

$$t \to \lambda^z t, \quad x \to \lambda x, \quad r \to \lambda r,$$

- Consistency (Taylor 2015): The metric + a massive vector bkgd

$$A = \sqrt{\frac{2}{z}(z-1)}\frac{L}{r^z}dt.$$

form a solution of the 3d gravity theory

$$S = \frac{1}{16\pi G}\int d^3x\sqrt{-g}\left(R + \frac{z^2+1}{2L^2} - \frac{F^2}{4} - \frac{z}{2L^2}A^2\right).$$

z-dependence of L

- In the literature, the gravity length scale L has been taken to be independent of z.
- Consider an interval of length l along the x-direction, the RT formula gives the entanglement entropy

$$S = \frac{L}{2l_p} \log \frac{l}{\epsilon}$$

This matches with our field theory result if

$$AdS/LFT: \quad L = (z-1)l_P, \qquad (2312.16284).$$

- This relation is different from the standard AdS/CFT

$$AdS/CFT: \quad L = \frac{2c}{3} l_P, \qquad (\text{Brown-Henneaux})$$

- The difference is because the RK vac is different from the CFT vac: CFT/RK vac is defined by the annihilation ops in the mom./coord. space.

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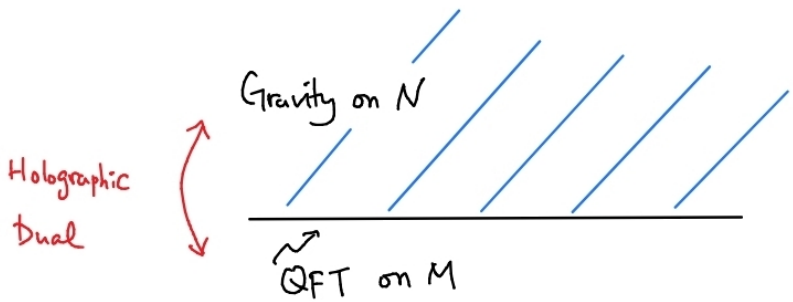
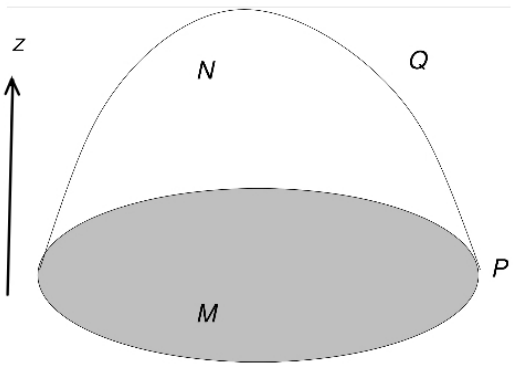
Variations of AdS holography

- Holography without conformal symmetry is even richer in physics.
- possibility 1 anisotrophy: AdS/LFT (Kachru 08; Taylor 15; 2312.16284).
- possibility 2 boundary: AdS/BCFT (Takayanagai 2011)
- Recently we considered a combination of boundary + anisotropy: AdS/BLFT.

AdS/BCFT

- Takayanagi proposed to extend that the gravity dual is given by gravity in a $d+1$ dimensional manifold N whose boundary is given by M and Q. (Takayanagi 2011)





- The bulk gravity action is given by

$$I = \int_N \sqrt{G}(R - 2\Lambda) + 2\int_Q \sqrt{h}(K - T)$$

T measures the boundary degrees of freedom (g-function).

- This gives the boundary variations

$$\delta I = \int_Q \sqrt{|h|} \left(K^{ij} - (K - T)h^{ij} \right) \delta h_{ij}$$

  and the NBC on Q: $K_{ij} - (K - T)h_{ij} = 0$ (Takayanagi 2011).
- Also possible is the CBC: $K = \frac{d}{d-1} T$ (Chu, Miao 2017).

They give the location of Q and both define consistent AdS/BCFT.

Proposal of AdS/BLFT

- Naively the bulk dual of BLFT is obtained by adding to

$$I_0 = \int_N d^{d+1}x\sqrt{-g}\left(R - 2\Lambda \underbrace{-\frac{F^2}{4} - \frac{1}{2}M^2A^2}_{\text{massive } A}\right)$$

the GH bdy term: $I_{GH} = 2\int_Q d^dx\sqrt{-h}(K-T)$.
However the BC $n_Q^M F_{MN} = 0$ is not satisfied for the bkgd of Taylor.

- A consistent BC can be obtained by adding the bdy action

$$I_A = \int_Q \sqrt{-h}\ \frac{1}{2}\mu M^2 a^2\ \cos\theta, \quad \text{(analogy of the GH bdy term)}$$

where a_μ is the projected gauge field on Q, θ is the angle between the normal of Q and the normal of M, μ some constant.

- This results in the modified BC for the vector field:

$$n_Q^N F_{N\alpha} + \mu M^2 \cos\theta \, a_\alpha = 0,$$

and the NBC and the CBC are modified to

$$\begin{aligned} \text{NBC}: \quad & K_{\alpha\beta} - (K - T)h_{\alpha\beta} = T^A_{\alpha\beta}, \\ \text{CBC}: \quad & d\,T - (d-1)K = T^A, \end{aligned}$$

$$T^A_{\alpha\beta} = \frac{1}{2}\mu M^2 \cos\theta \left(a_\alpha a_\beta - \frac{1}{2} h_{\alpha\beta} a^2 \right), \quad T^A = \left(1 - \frac{d}{2}\right) \frac{1}{2} \mu M^2 \cos\theta \; a^2$$

- For the example of 2d BLFT defined on the half space $x \geq 0$, we find the soln

$$x = r \sinh\left(\frac{\rho_*}{L}\right),$$

It is independent of the anisotropy. It is remarkable that the same brane profile universally defines the 2d BLFT.

g-theorem for BCFT and BLFT

- Generally given a BCFT, one can consider a perturbation to a BCFT at the UV by a set of relevant local operators $\{\phi_i\}$ at the boundary,

$$S = S_{BCFT_{UV}} + \int_{\partial M} \lambda_i \phi_i.$$

  The g-theorem asserts the existence of a g-function which decreases monotonically along the boundary RG flow, with value given by the boundary central charge of the BCFT at the RG fixed point.

- For 2d BCFT, g-theorem was proven by Friedan 2003. and for holographic BCFT by Takayaganai 2011: boundary perturbation modeled by a perturbation $T^Q_{\mu\nu}$ on Q, and g-theorem follow from the imposition of the null energy condition on Q.

- For BLFT, field theoretic proof is absent. We employed AdS/BLFT and establised a version of the g-theorem (2409.06667): needs NEC or DEC

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Lifshitz QM

- To see more clearly the quantum effect of anisotropy, we consider QM.
- Consider an anisotropic media with dispersion relation $\omega \sim k^z$ for the noninteracting quanta. Add a potential, we obtain the *Lifshitz quantum mechanics* defined by

$$i\hbar\frac{\partial}{\partial t}\psi = H\psi, \quad H = -\frac{\hbar^z}{M^{z-2}}\frac{\Delta_z}{2m} + V.$$

M is a mass scale associated with the anisotropy.

Local conservation of probability

- Also refereed to as fractional quantum mechanics. However, local probability is violated with a source term? e.g. Laskin, Principles of fractional QM, World Scientific, 2011.
- We find that the local conservation (Chu-Gunawan 2507.14996)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot \mathbf{J} = 0,$$

is observed with $\rho = \psi^* \psi$ and

$$\mathbf{J} = \lambda_M^{z-2} \frac{\hbar}{2mi} \frac{\nabla_x - \nabla_{x'}}{\nabla_x^2 - \nabla_{x'}^2} \left[(\nabla_x^2)^{z/2} - (\nabla_{x'}^2)^{z/2} \right] \psi^*(\mathbf{x}', t) \psi(\mathbf{x}, t) \Big|_{\mathbf{x}'=\mathbf{x}},$$

where $\lambda_M := \hbar/M$ is a length scale associated with the anisotropy.

Q. How to observe this anisotropic effect on $\mathbf{J}$?
$\longrightarrow$ Tunneling current in JJ!

Anisotropic Josephson Junction

- Consider a SIS-type JJ made up of two superconductors separated by a thin anisotropic insulator. e.g. by material engineering to achieve the desired dispersion relation
- The superconductors have Landau-Ginzburg order parameters

$$\psi_i = \sqrt{n_i} e^{\mathrm{i}\theta_i}, \quad i = 1, 2$$

that can be interpreted as the wave functions of the Cooper pairs.

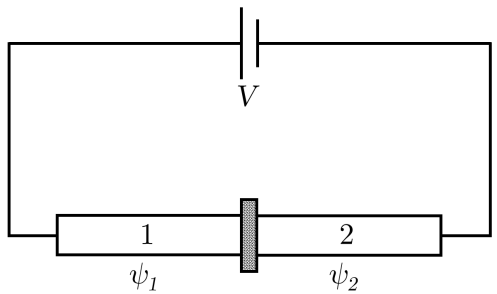

Figure: Two superconductors separated by a thin insulator

- Wave equation inside the insulator,

$$\lambda_M^{z-2}\frac{\hbar^2\Delta_z\psi}{2m} = (V_0 - E)\psi$$

has solution

$$\psi = C_1 \cosh\frac{x}{\zeta} + C_2 \sinh\frac{x}{\zeta},$$

$$\zeta := \zeta_0\left(\frac{\zeta_0}{\lambda_M}\right)^{2/z-1}, \quad \text{with } \zeta_0 := \frac{\hbar}{\sqrt{2m(V_0 - E)}}.$$

- ζ is the tunneling penetration length and ζ_0 is the tunneling penetration length in the undeformed $z = 2$ (isotropic) case.
- We obtain the probability current

$$J = \frac{z\zeta}{2\zeta_0}\cdot\frac{\hbar}{m\zeta_0}\mathrm{Im}(C_1^* C_2).$$

- The coefficients C_1, C_2 can be determined by imposing the boundary conditions at $x = \pm a$:

$$\psi(-a) = \sqrt{n_1} e^{i\theta_1}, \quad \psi(a) = \sqrt{n_2} e^{i\theta_2}.$$

- As a result, we obtain

$$\text{first Josephson equation:} \quad J = J_c \sin\varphi, \quad \varphi = \theta_2 - \theta_1$$

for the tunneling current. Here the critical current $J_c = \gamma J_{c0}$ with

$$J_{c0} = \frac{q\hbar\sqrt{n_1 n_2}}{m\zeta_0} \frac{1}{\sinh 2a/\zeta_0}, \quad \gamma = \frac{z\zeta}{2\zeta_0} \frac{\sinh 2a/\zeta_0}{\sinh 2a/\zeta},$$

where J_{c0} is the critical current in the isotropic case and γ is an modification factor due to anisotropy.

- With the application of a voltage V across the JJ, the phase becomes time dependent ,

$$\text{second Josephson equation:} \quad \frac{\partial\varphi}{\partial t} = \frac{q}{\hbar} V.$$

this result is unmodified by anisotropy.

- In the usual isotropic case, the critical current J_{c0} is exponentially suppressed by a factor of $e^{-2a/\zeta}$ since the Cooper pairs cannot tunnel deep enough into the insulating region.
- The Josephson current can be enhanced ($\gamma > 1$) if the penetration length is made bigger ($\zeta > \zeta_0$). This happens if $z < 2$ and $\lambda_M < \zeta_0$ or $z > 2$ and $\lambda_M > \zeta_0$.
  e.g. $z = 0.8$ and $\zeta_0/\lambda_M = 10$, we can have a huge amplification factor $\gamma \sim 10^5$; or with $z = 5$ and $\zeta_0/\lambda_M = 1/10$, we have $\gamma \sim 10^4$.

The amplification maybe useful for improving the performance of JJ

Summary

We have developed new areas of research on quantum system with Lifshitz anisotropy.

- LFT
  - construction of LFT for arbitrary z
  - family of degenerate RK vacuum
  - entanglement entropy allows an refinement of AdS/LFT: z-dependence of L
- New proposal of AdS/BLFT
  - g-theorem
  - novel behaviour of entanglement entropy in the presence of anisotropy and boundary
- Anisotropic QM
  - unitraity established
  - anisotropic amplification of JJ