Quantum Geometry and Linear and Nonlinear Responses of Solids

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Fundamental Quantum Science Research Program (FQSP)
RIKEN Center for Emergent Matter Science (CEMS)

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and many others

Quantum Geometry Phenomena in Condensed Matter Systems

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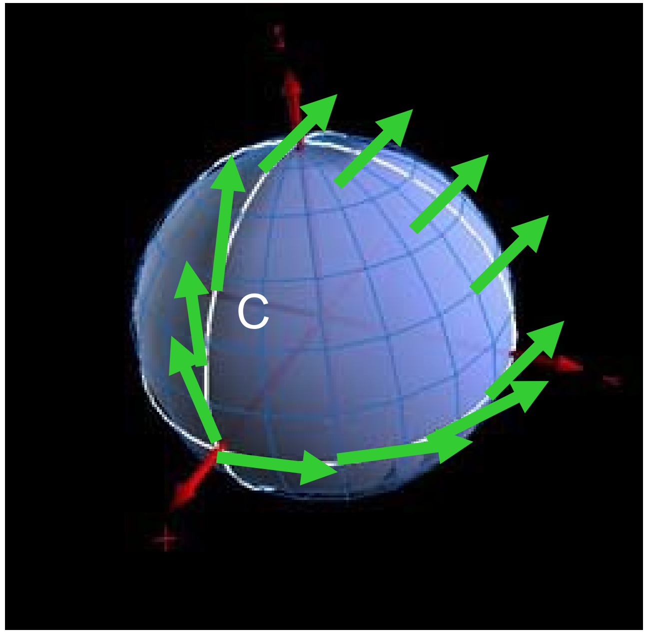
arXiv:2508.00469

Any comments are welcome !

Plan of this talk

1. Geometry in quantum mechanics - Berry curvature and metric
2. Band structure as manifold in Hilbert space
   - quantum geometric tensor of Bloch wave function
3. Linear response and quantum geometry
4. Nonlinear and nonreciprocal responses in quantum materials
5. Quantum metric in flat-band superconductors
6. Nonlinear optics as quantum geometric phenomena
7. Summary and perspectives

Manifold in Hilbert space



Subspace of Hilbert space parametrized by
the parameter set $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$

Often by the energy eigenvalues separated by gap

This subspace is regarded as a manifold, and can be analyzed by differential geometry and topology.

Berry connection and Berry curvature

M.V.Berry, Proc. R.Soc. Lond. A392, 45(1984)

$|u(\lambda)\rangle$ depends on the parameter set $\lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n)$

$$\langle u(\lambda)|u(\lambda + d\lambda)\rangle = 1 + \sum_i \langle u(\lambda)|\partial_i|u(\lambda)\rangle d\lambda_i .$$

$$\langle \partial_i u|u\rangle + \langle u|\partial_i u\rangle = 2\mathrm{Re}[\langle u|\partial_i u\rangle] = 0$$

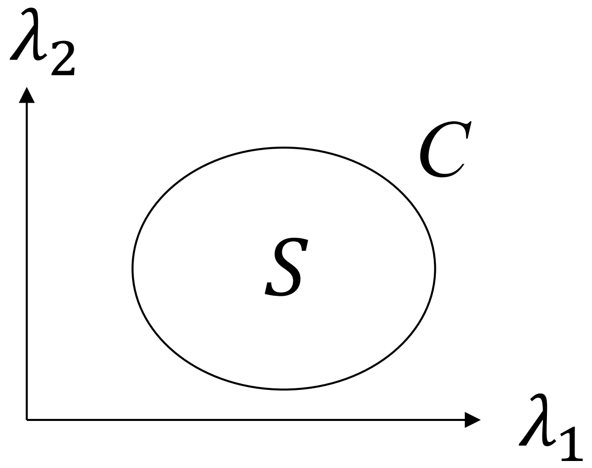
$$a_i(\lambda) = i\langle u(\lambda)|\partial_i|u(\lambda)\rangle$$ Berry connection

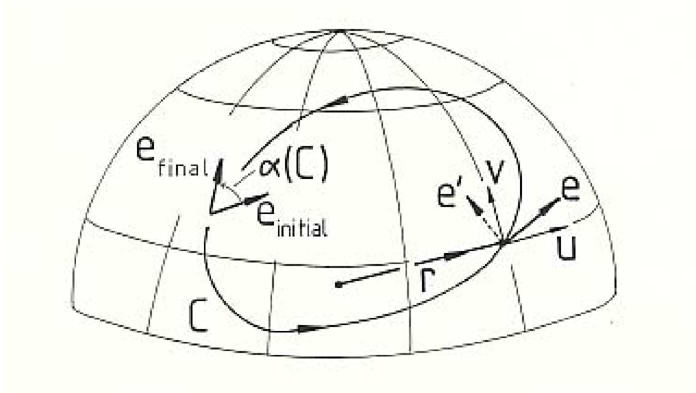
$$\langle u(\lambda)|u(\lambda + d\lambda)\rangle = \exp[-i\sum_i a_i(\lambda)d\lambda_i]$$

$$a_i(\lambda) \to a_i(\lambda) - \partial_i\phi(\lambda)$$ gauge trans.

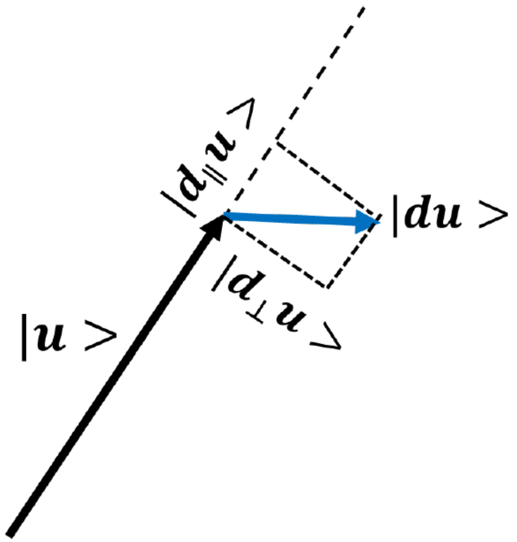
$$F_{ij}(\lambda) = \partial_i a_j(\lambda) - \partial_j a_i(\lambda)$$ Berry curvature

$$\oint_C d\lambda \cdot a = \iint_S dS \cdot F$$





Quantum metric



$$|d_{\|} u(\lambda)\rangle = |u(\lambda)\rangle\langle u(\lambda)|du(\lambda)\rangle,$$
$$|d_{\perp} u(\lambda)\rangle = |du(\lambda)\rangle - |u(\lambda)\rangle\langle u(\lambda)|du(\lambda)\rangle$$

$$ds^2 = ||d_{\perp} u(\lambda)\rangle||^2 = \sum_{i,j} g_{ij}(\lambda) d\lambda_i \lambda_j$$

distance between the two neighboring w.f.

$$g_{ij}(\lambda) = \mathrm{Re}[\langle \partial_i u(\lambda)|\partial_j u(\lambda)\rangle] - a_i(\lambda) a_j(\lambda)$$

quantum metric tensor

Quantum Geometric Tensor

$|u(\lambda)\rangle$ Quantum state which depends on the parameter set $\boldsymbol{\lambda} = (\lambda_1, \cdots, \lambda_n)$

$P(\lambda) = |u(\lambda)\rangle\langle u(\lambda)|$ Projection operator - gauge invariant

$G_{ij}(\lambda) = \mathrm{Tr}[P(\lambda)\partial_i P(\lambda)\partial_j P(\lambda)]$ Quantum geometric tensor

$G_{ij}(\lambda) = \langle \partial_i u(\lambda)|\partial_j u(\lambda)\rangle - a_i(\lambda)a_j(\lambda)$ $\quad a_i(\lambda) = i\langle u(\lambda)|\partial_i|u(\lambda)\rangle$ Berry connection

$$G_{ij}(\lambda) = g_{ij}(\lambda) - \frac{i}{2}F_{ij}(\lambda)$$

$$ds^2 = ||d_\perp u(\lambda)\rangle||^2 = \sum_{i,j} g_{ij}(\lambda)d\lambda_i\lambda_j$$

Quantum distance between the neighboring states → quantum metric

$F_{ij}(\lambda) = \partial_i a_j(\lambda) - \partial_j a_i(\lambda)$ Berry curvature

$\boldsymbol{\eta_i^* G_{ij} \eta_j \geq 0}$ Positive definite

Quantum geometric tensor of an eigenstate of Hamiltonian

$$H(\lambda)|u_k(\lambda)\rangle = E_k(\lambda)|u_k(\lambda)\rangle$$

$$[\partial_i H(\lambda)]|u_k(\lambda)\rangle + H(\lambda)|\partial_i u_k(\lambda)\rangle$$
$$= [\partial_i E_k(\lambda)]|u_k(\lambda)\rangle + E_k(\lambda)|\partial_i u_k(\lambda)\rangle$$

Feynman-Hellmann theorem

$$\langle u_k(\lambda)|\partial_i H(\lambda)|u_k(\lambda)\rangle = \partial_i E_k(\lambda)$$

$$\langle u_l(\lambda)|\partial_i u_k(\lambda)\rangle = \frac{\langle u_l(\lambda)|\partial_i H(\lambda)|u_k(\lambda)\rangle}{E_k(\lambda) - E_l(\lambda)}$$

$$G_{k,ij}(\lambda) = \langle \partial_i u_k|(1 - |u_k\rangle\langle u_k|)|\partial_j u_k\rangle$$
$$= \sum_{m(\neq k)} \langle \partial_i u_k|u_m\rangle\langle u_m|\partial_j u_k\rangle$$
$$= \sum_{m(\neq k)} \frac{\langle u_k|\partial_i H|u_m\rangle\langle u_m|\partial_j H|u_k\rangle}{(E_k - E_m)^2}.$$

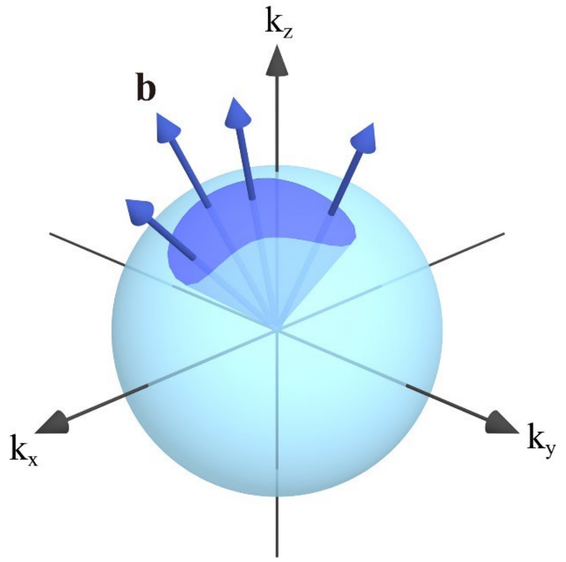
$$H = h_0(\lambda) + \vec{h}(\lambda)\cdot\vec{\sigma}$$ 2x2 Hamiltonian

$$G^{\pm}_{ij} = \frac{1}{4}\partial_i\vec{n}\cdot\partial_j\vec{n} \pm \frac{i}{4}\vec{n}\cdot(\partial_i\vec{n}\times\partial_j\vec{n})$$

$$\vec{n}(\lambda) = \vec{h}(\lambda)/|\vec{h}(\lambda)|$$

$$G \sim 1/h(\lambda)^2$$

Monopole singularity at $h(\lambda) = 0$



Linear response and quantum geometry

$$\sigma_{\mu\nu}(\omega) = \frac{ine^2}{m\omega_+}\delta_{\mu\nu} + \frac{1}{\omega_+}\int_0^t dt e^{i\omega_+ t}\langle[\hat{J}_\mu(t), \hat{J}_\nu(0)]\rangle$$

$$= \frac{ine^2}{m\omega_+}\delta_{\mu\nu} + \frac{i}{\omega_+}\sum_{n,m}\int(dk)\frac{f_{nm}(J_\mu)_{nm}(J_\nu)_{mn}}{\omega_+ + \omega_{nm}}$$

Kubo formula

$$\sigma_{\mu\nu}(\omega) = i\sum_{n,m}\int(dk)\frac{f_{nm}}{\omega_{nm}}(J_\mu)_{nm}(J_\nu)_{mn}$$

$$\times\left(\frac{1}{\omega_+ + \omega_{nm}} - \frac{1}{\omega_{nm}}\right)$$

$$= -i\sum_{n,m}\int(dk)\frac{f_{nm}}{\omega_{nm}}\frac{(J_\mu)_{nm}(J_\nu)_{mn}}{\omega_+ + \omega_{nm}}$$

$$= \sigma^{\text{intra}}_{\mu\nu}(\omega) + \sigma^{\text{inter}}_{\mu\nu}(\omega)$$

$\boldsymbol{n = m}$ $\boldsymbol{n \neq m}$

$$\sigma^{\text{inter}}_{\mu\nu}(0) = -i\sum_{n\neq m}\int(dk)f_{nm}\frac{(J_\mu)_{nm}(J_\nu)_{mn}}{\omega^2_{nm}}$$

$$= -ie^2\sum_{n}\int(dk)f_n([x_\mu, x_\nu])_{nn}$$

$$= -e^2\sum_{n}\int(dk)f_n F_{n,\mu\nu}$$

TKNN formula

Linear response and quantum geometry - Sum rule

$$\sigma^{\text{abs.}}_{\mu\nu}(\omega) = -\pi \sum_{n,m} \int (dk) \frac{f_{nm}}{\omega_{nm}} (J_\mu)_{nm} (J_\nu)_{mn} \delta(\omega + \omega_{nm})$$

Absorptive part of conductivity

$$= -i\frac{\pi e^2}{m} \sum_{n,m} \int (dk) f_{nm} (x_\mu)_{nm} (p_\nu)_{mn} \delta(\omega + \omega_{nm})$$

$$\langle \psi_{mk} | \hat{x} | \psi_{nk'} \rangle = i\delta_{m,n} \frac{\partial \delta_{k,k'}}{\partial k} + i \langle u_{mk} | \frac{\partial}{\partial k} u_{nk} \rangle \delta_{k,k'}$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \sigma^{\text{abs.}}_{\mu\nu}(\omega)$$

$$= -\frac{ie^2}{2m} \sum_{n,m} \int (dk) f_{nm} (x_\mu)_{nm} (p_\nu)_{mn}$$

$$= -\frac{ie^2}{2m} \sum_{n,m} \int (dk) f_n [(x_\mu)_{nm} (p_\nu)_{mn} - (x_\mu)_{mn} (p_\nu)_{nm}]$$

$$= -\frac{ie^2}{2m} \sum_{n} \int (dk) f_n [x_\mu, p_\nu]_{nn}$$

$$= \frac{e^2}{2m} \delta_{\mu\nu} \sum_{n} \int (dk) f_n$$

$$= \frac{e^2 n}{2m} \delta_{\mu\nu},$$

$$\int_{-\infty}^{\infty} \frac{d\omega}{2\pi} \frac{\sigma^{\text{abs.}}_{\mu\nu}(\omega)}{\omega}$$

$$= \frac{e^2}{2m} \sum_{n,m} \int (dk) f_{nm} (x_\mu)_{nm} (x_\nu)_{mn}$$

$$= \frac{e^2}{2m} \sum_{n \neq m} \int (dk) f_n [(x_\mu)_{nm} (x_\nu)_{mn} - (x_\mu)_{mn} (x_\nu)_{nm}]$$

$$= \frac{e^2}{2m} \sum_{n} \int (dk) f_n (\langle \partial_\mu n | \partial_\nu n \rangle - \langle \partial_\mu n | n \rangle \langle n | \partial_\nu n \rangle)$$

$$= \frac{e^2}{2m} G_{\mu\nu},$$

For insulator $\omega \geq E_g$ $\Rightarrow$ $G_{\mu\nu} \geq \frac{n}{E_g} \delta_{\mu\nu}$

Quantum geometrical bound relations for observables

Koki Shinada[1,*] and Naoto Nagaosa[1,2,†]

arXiv:2507.12836

External field	Conjugate operator
vector potential A	electric current $\hat{J}$
scalar potential ϕ	electric density $\hat{n}$
electric field E	electric polarization $\hat{P}_e$
magnetic field B	spin $\hat{S}$
strain tensor ∂u	stress tensor $\hat{\sigma}$

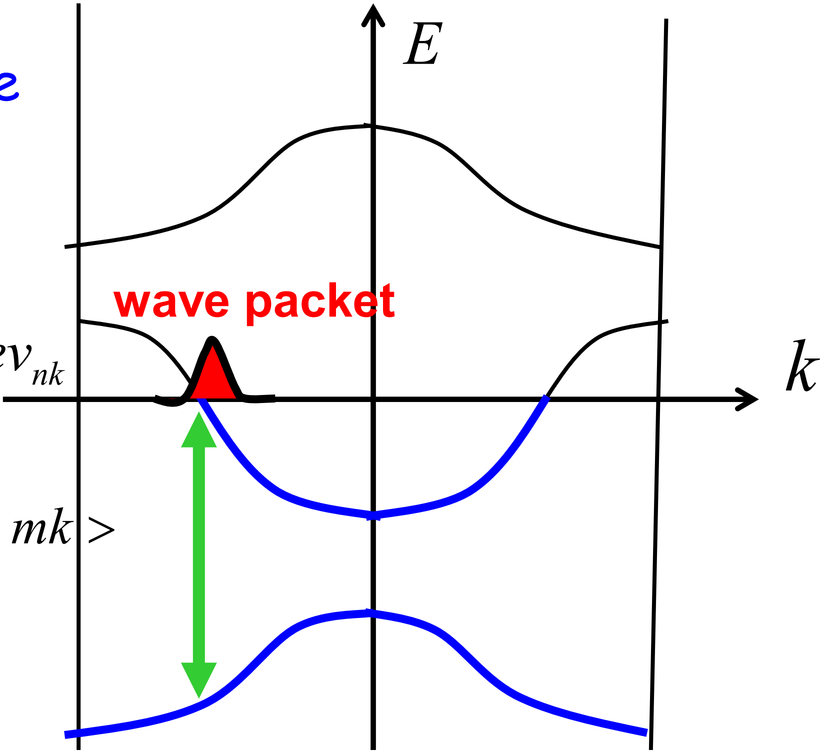
$$[A]^{\beta} = \int_0^{\infty} A(\omega)/\omega^{\beta} d\omega.$$

	A	E	B	u	∂u
A	ω_{p}, $[\mathrm{Im}\sigma^{(\mathrm{AS})}]^0$	σ_{Hall} or 0, $[\mathrm{Re}\sigma^{(\mathrm{S})}]^1$	0, $[\mathrm{Im}\chi^{\mathrm{me(S)}}]^0$	χ_{Au}, $[\chi_{Au}^{(\mathrm{AS})}]^1$	$\chi_{A\partial u}$, $[\chi_{A\partial u}^{(\mathrm{AS})}]^1$
E		χ^{e}, $[\mathrm{Im}\sigma^{(\mathrm{AS})}]^2$	$\chi^{\mathrm{me(S)}}$, $[\mathrm{Re}\chi^{\mathrm{me(AS)}}]^1$	Z, $[\mathrm{Re}Z^{(\mathrm{AS})}]^1$	d, $[\mathrm{Re}d^{(\mathrm{AS})}]^1$
B			χ^{m}, $[\mathrm{Re}\chi^{\mathrm{m(AS)}}]^1$	Z^{m}, $[\mathrm{Re}Z^{\mathrm{m(AS)}}]^1$	Q, $[\mathrm{Re}Q^{(\mathrm{AS})}]^1$
u				Z^{u}, $[\mathrm{Re}Z^{\mathrm{u(AS)}}]^1$	$\chi_{u\partial u}$, $[\chi_{u\partial u}^{(\mathrm{AS})}]^1$
∂u					λ, $[\mathrm{Re}\lambda^{(\mathrm{AS})}]^1$

Intra- and Inter-band matrix elements of current

Uncertainty principle
$\Delta x \Delta k > 1$

$$< nk \,|\, J \,|\, nk > = -e v_{nk}$$



$$< nk \,|\, J \,|\, mk >$$

Wavefunction matters !!

Of-diagonal matrix elements

- Distort the band structure by E
- Additional current

Even a filled band can
support current
e.g., polarization current
quantum Hall current

$$< nk \,|\, J \,|\, nk > = -e < nk \,|\, \frac{\partial H(k)}{\partial k} \,|\, nk > = -e \frac{\partial \varepsilon_n(k)}{\partial k}$$

$$< nk \,|\, J \,|\, mk > = -e < nk \,|\, \frac{\partial H(k)}{\partial k} \,|\, mk > = -e(\varepsilon_{nk} - \varepsilon_{mk}) < nk \,|\, \frac{\partial}{\partial k} \,|\, mk >$$

Time-reversal	$b(k) = -b(-k)$
Inversion	$b(k) = b(-k)$
➡	$b(k) = 0$

Berry Phase in k-space

$$\psi_{nk}(r) = e^{ikr} u_{nk}(r)$$

Bloch wavefucntion

$$a_n(k) = -i < u_{nk} | \nabla_k | u_{nk} >$$

Berry phase connection in k-space

$$x_i = r_i + a_n(k) = i\partial_{k_i} + a_n(k)$$

covariant derivative

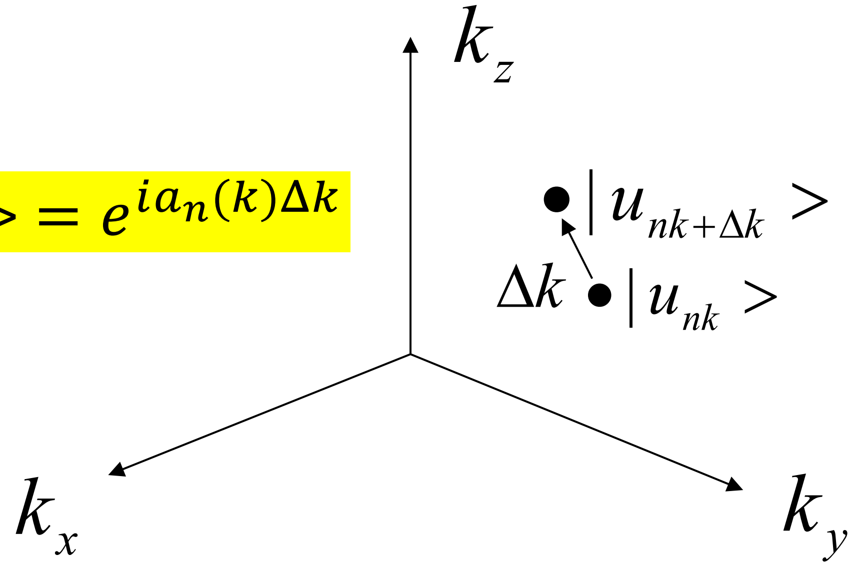
$$[x, y] = i(\partial_{k_x} a_{ny}(k) - \partial_{k_y} a_{nx}(k)) = ib_{nz}(k)$$

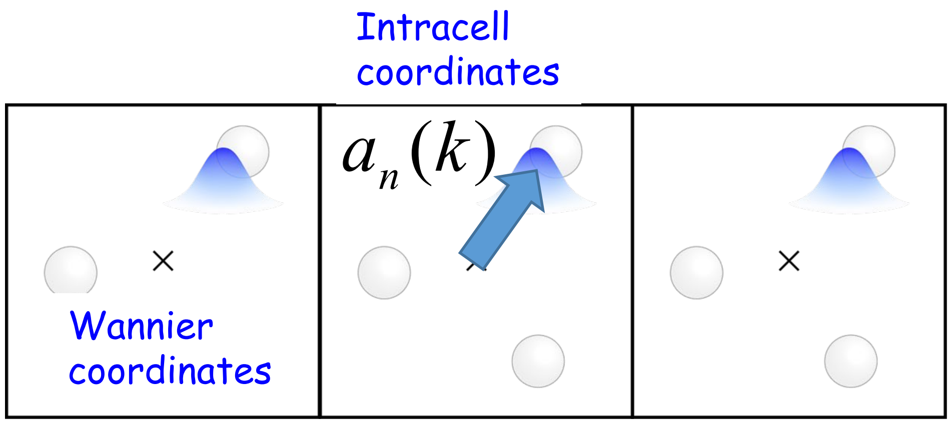
Curvature in k-space

$$\frac{dx(t)}{dt} = -i[x, H] = \frac{k_x}{m} - i[x, y]\frac{\partial V}{\partial y} = \frac{k_x}{m} + b_{nz}(k)\frac{\partial V}{\partial y}$$

Anomalous Velocity and
Anomalous Hall Effect

$$<u_{nk}|u_{nk+\Delta k} > = e^{ia_n(k)\Delta k}$$





Correct equation of motion taking into account inter-band matrix element

$$\frac{d\vec{r}(t)}{dt} = \frac{\partial \varepsilon_n(\vec{k})}{\partial \vec{k}} - \vec{B}_n(\vec{k}) \times \frac{d\vec{k}(t)}{dt}$$

k-space curvature

anomalous velocity

Luttinger, Blount, Niu

$$\frac{d\vec{k}(t)}{dt} = -\frac{\partial V(\vec{r})}{\partial \vec{r}} - \vec{B}(\vec{r}) \times \frac{d\vec{r}(t)}{dt}$$

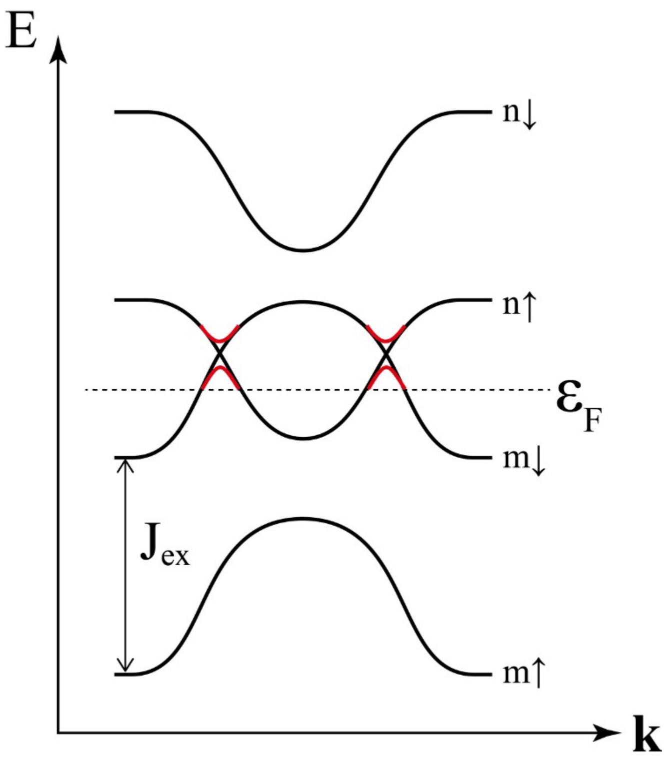
r-space curvature

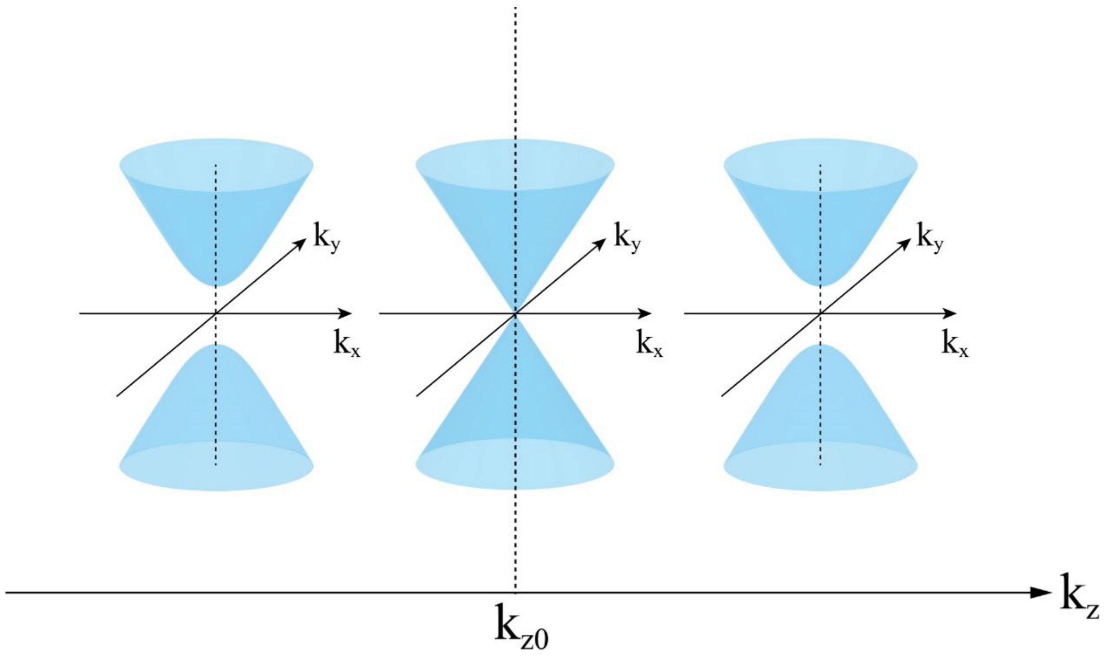
Origin of the k-space curvature = interband current matrix

$$B_n(k) = \nabla \times A_n(k) \qquad A_n(k) = i < nk \,|\, \nabla \,|\, nk >$$

How the wavefunction is connected in k-space
→ Berry phase

Band crossings and magnetic monopoles in magnets





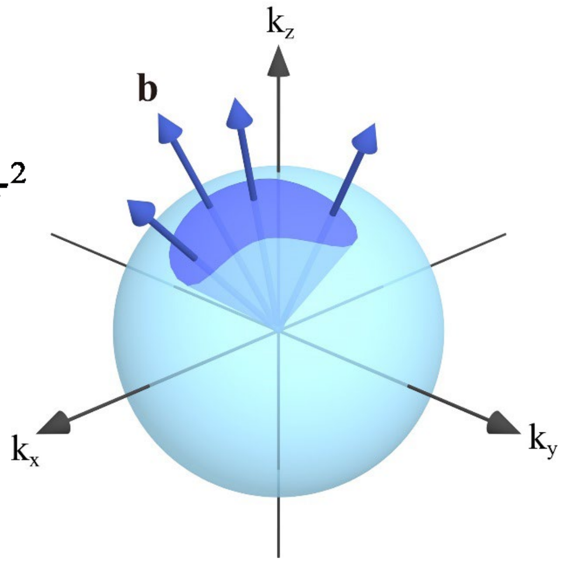
$$H = \frac{\varepsilon_{n\uparrow}(\mathbf{k}) + \varepsilon_{m\downarrow}(\mathbf{k})}{2} + \frac{\varepsilon_{n\uparrow}(\mathbf{k}) - \varepsilon_{m\downarrow}(\mathbf{k})}{2}\sigma^3 + \lambda_1(\mathbf{k})\sigma^1 + \lambda_2(\mathbf{k})\sigma^2$$

$$H = \eta v(\mathbf{k} - \mathbf{k}_0)\cdot\sigma$$

Weyl fermion

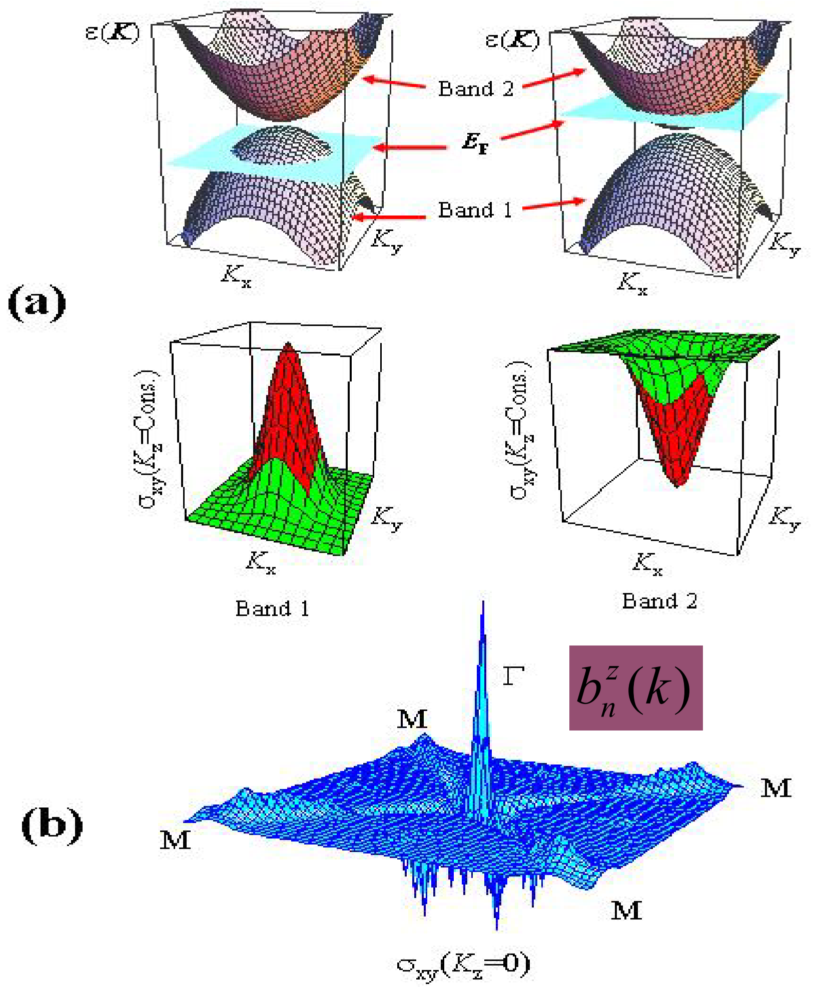
$$\mathbf{b}(\mathbf{k}) = \frac{\eta(\mathbf{k} - \mathbf{k}_0)}{2|\mathbf{k} - \mathbf{k}_0|^3}$$

Magnetic monopole
in momentum space

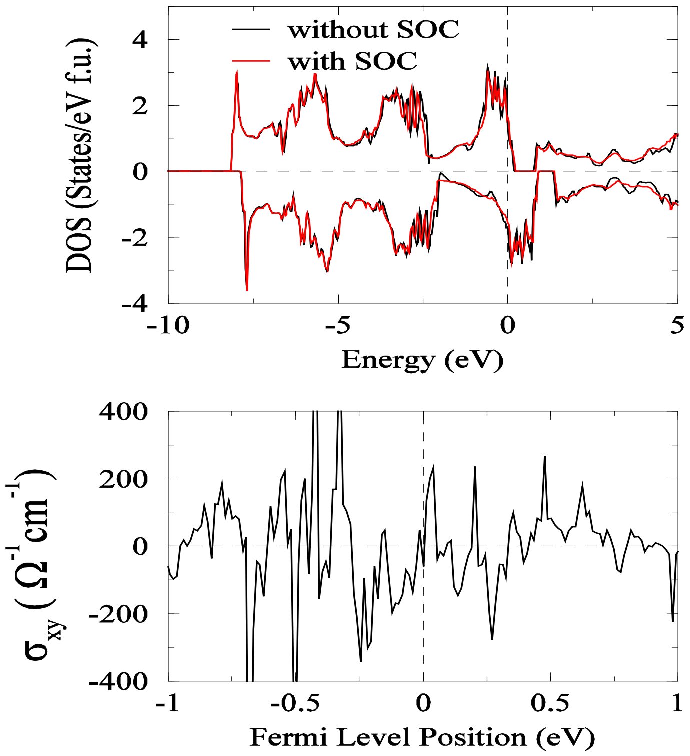


Anomalous Hall Effect in SrRuO3 - Magnetic Monopole in k-Space

Z.Fang et al. Science2003

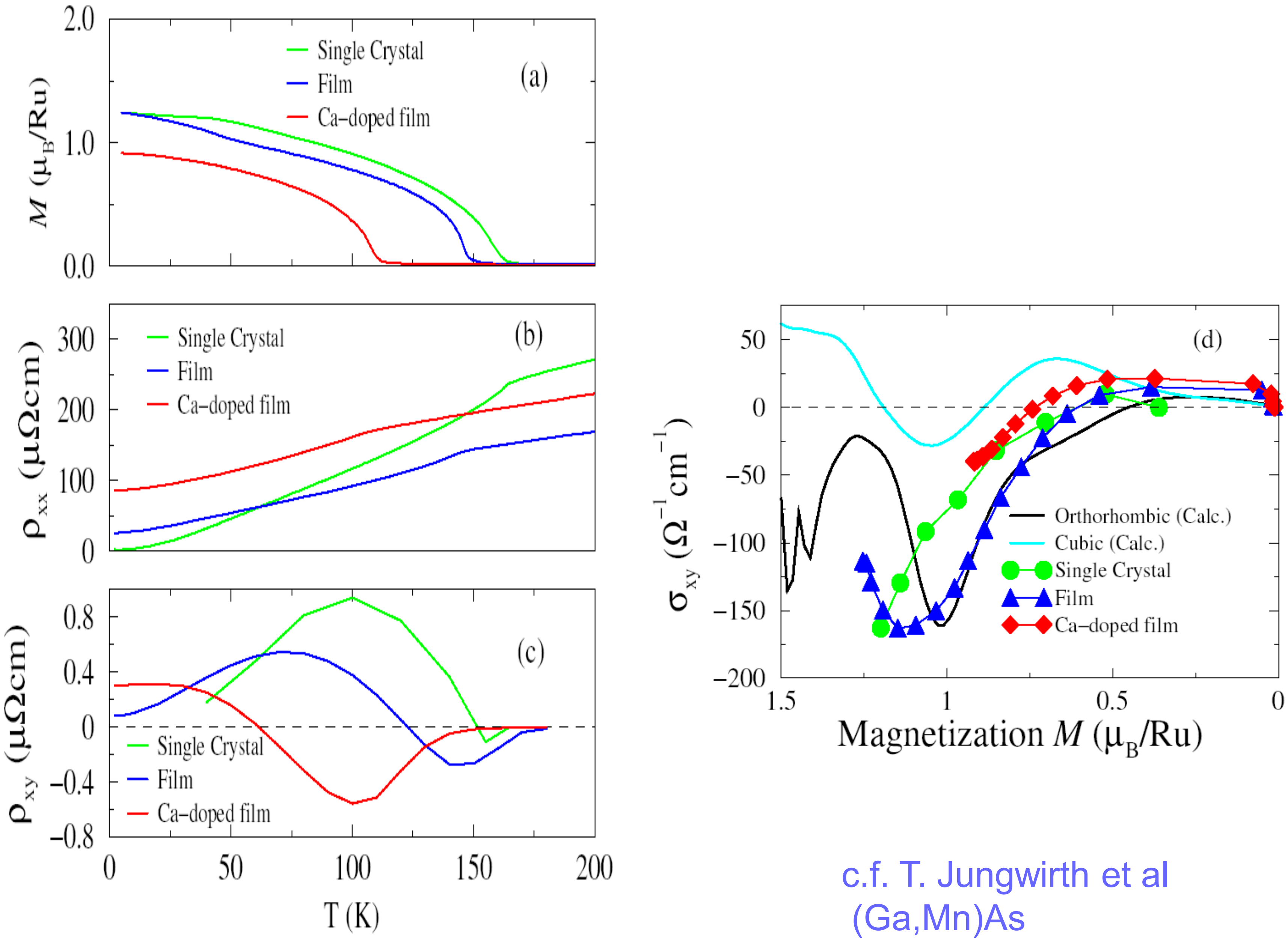


$b_n^z(k)$



$$\sigma_{ij}^{TKNN} = -\epsilon_{ij\ell} e^2 \hbar \sum_n \int \frac{d^d \mathbf{p}}{(2\pi\hbar)^2} b_n^\ell(\mathbf{p}) f(\varepsilon_n(\mathbf{p}))$$

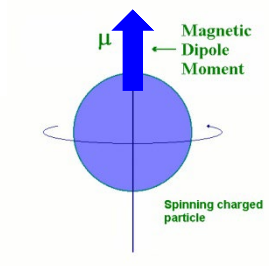
Small energy scale 0.02eV
Behavior like quantum chaos



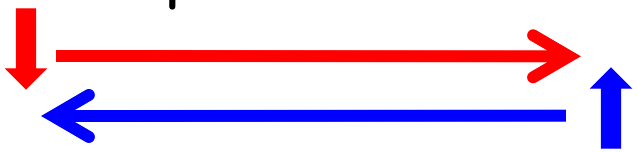
c.f. T. Jungwirth et al
(Ga,Mn)As

Classification of Order Parameters

Time reversal / Inversion	even	odd
even	ρ charge density	Spin $\vec{M}$ magnetization
odd	$\vec{j}_s$, $\vec{P}$ **spin current** polarization	$\vec{j}$, $\vec{T}$ current **toroidal moment**



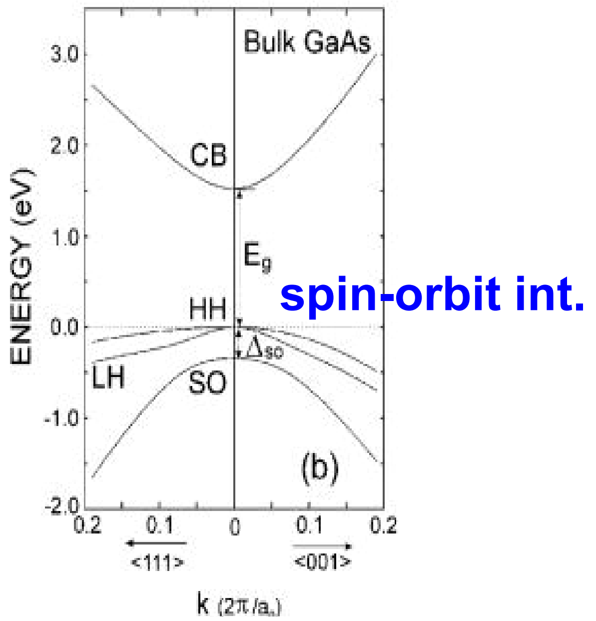
Spin current



Spin Hall effect in semiconductors

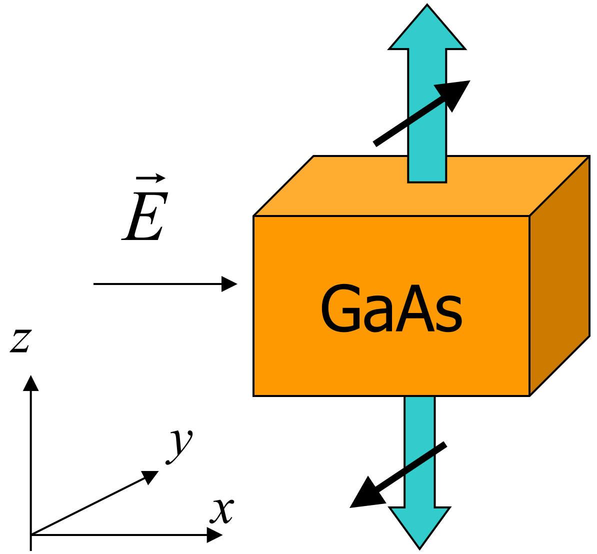
$$j_{xy} = \frac{eE_z}{12\pi^2\hbar}\left(k_F^H - k_F^L\right) \equiv \frac{1}{2e}\sigma_s E_z$$

x: current direction
y: spin direction
z: electric field



SU(2) analog of the QHE
- topological origin
- dissipationless
- All occupied states in the valence band contribute.

External electric field does not break time-reversal symmetry.
Spin current is allowed in this system with time-reversal symmetry



Wave-packet formalism in systems with Kramers degeneracy

Let us extend the wave-packet formalism to the case with time-reversal symmetry.

Adiabatic transport

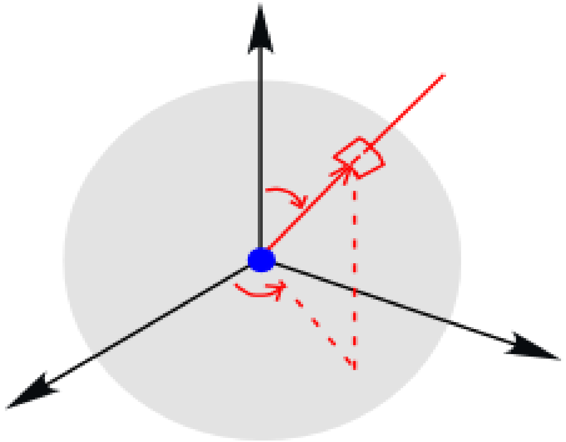
= The wave-packet stays in the same band, but can transform inside the Kramers degeneracy.

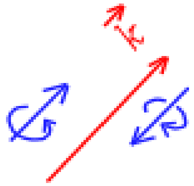
$$\left|\psi_n(t)\right\rangle = \int d^3q \,\left(a_1(\vec{q},t)\left|\psi_{n1}(\vec{q},\vec{x}_c,t)\right\rangle + a_2(\vec{q},t)\left|\psi_{n2}(\vec{q},\vec{x}_c,t)\right\rangle\right) \quad (n = H, L)$$

$$\begin{pmatrix} z_1(\vec{q},t) \\ z_2(\vec{q},t) \end{pmatrix} = \frac{1}{\sqrt{a_1^2 + a_2^2}} \begin{pmatrix} a_1(\vec{q},t) \\ a_2(\vec{q},t) \end{pmatrix}$$

Eq. of motion

$$\begin{cases} \dot{\vec{k}} = -e\vec{E} \\ \dot{x}_l = \dfrac{\partial E^n}{\partial k_l} - \dot{\vec{k}}_j\left(z^+ F_{lj}^n z\right) \qquad n = H, L \\ \dot{z} = i\left(\dot{\vec{k}} \cdot \vec{A}^n\right) z \end{cases}$$

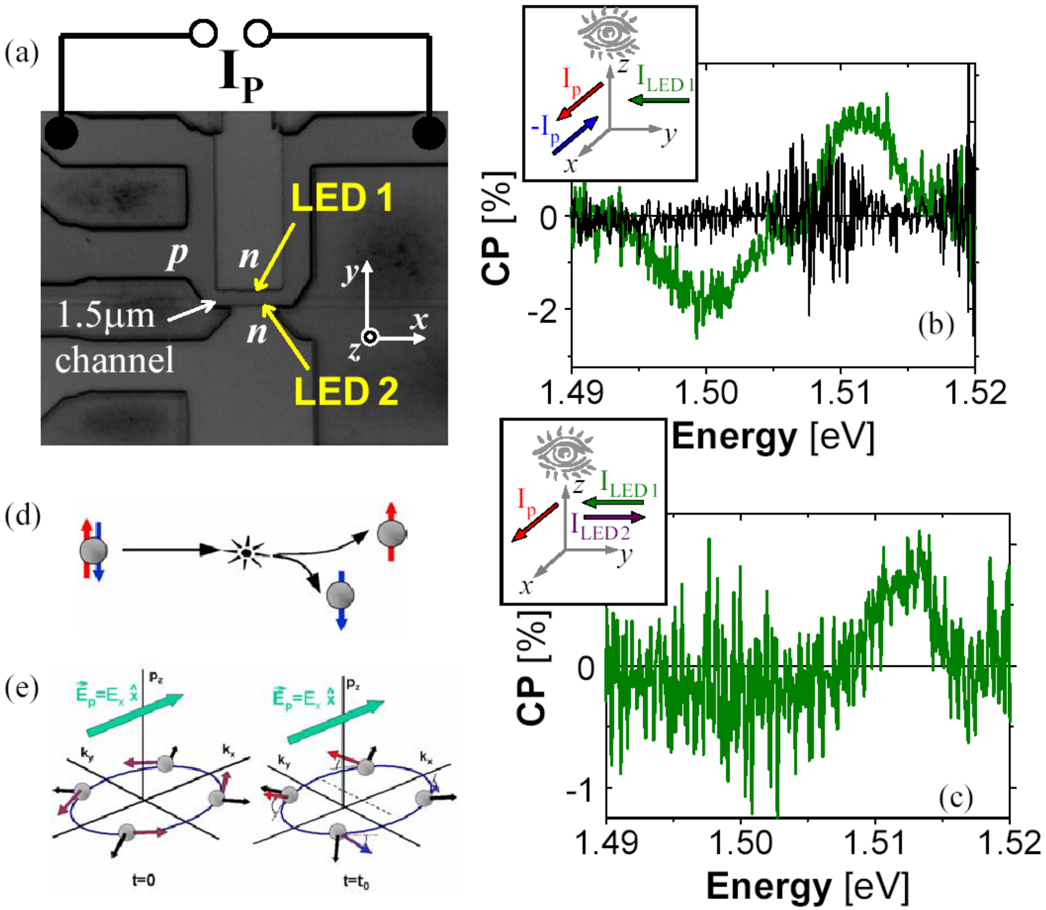




Experimental confirmation of spin Hall effect in GaAs

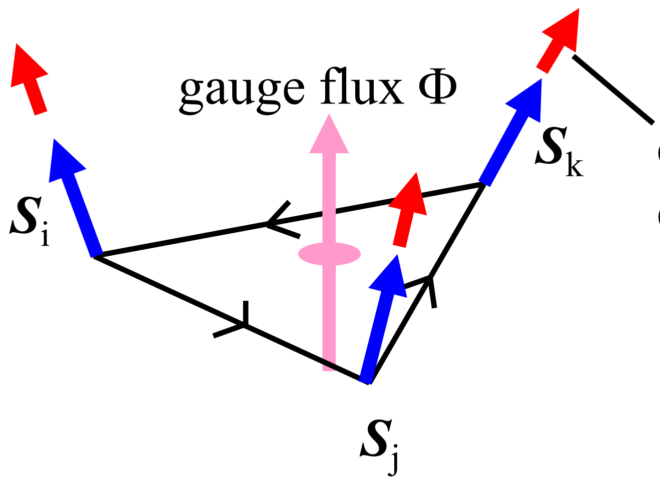
D.D.Awschalom (n-type) UC Santa Barbara

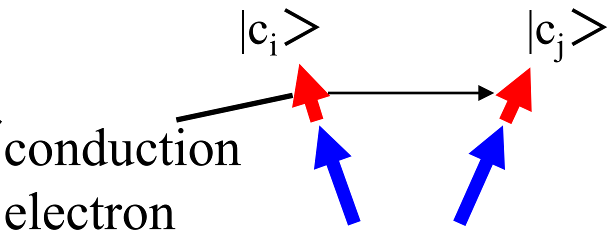
J.Wunderlich (p-type) Hitachi Cambridge



Wunderlich et al. 2004

Solid angle by spins acting as a gauge field





$$\begin{aligned} t_{ij} &= t\langle\chi_j|\chi_i\rangle \\ &= t\left(\cos\frac{\theta_i}{2}\cos\frac{\theta_j}{2}+\sin\frac{\theta_i}{2}\sin\frac{\theta_j}{2}\exp(i(\phi_j-\phi_i))\right) \\ &= t\cos\frac{\theta_{ij}}{2}\exp(ia_{ij}) \end{aligned}$$

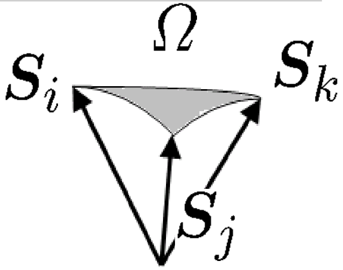
$b_{eff} \approx 4000T$ for nm scale

acquire a phase factor = Berry phase

Fictitious flux (in a continuum limit)

$$\Phi \propto \frac{\mathbf{S}_i\cdot(\mathbf{S}_j\times\mathbf{S}_k)}{2} = \frac{\Omega}{2}$$

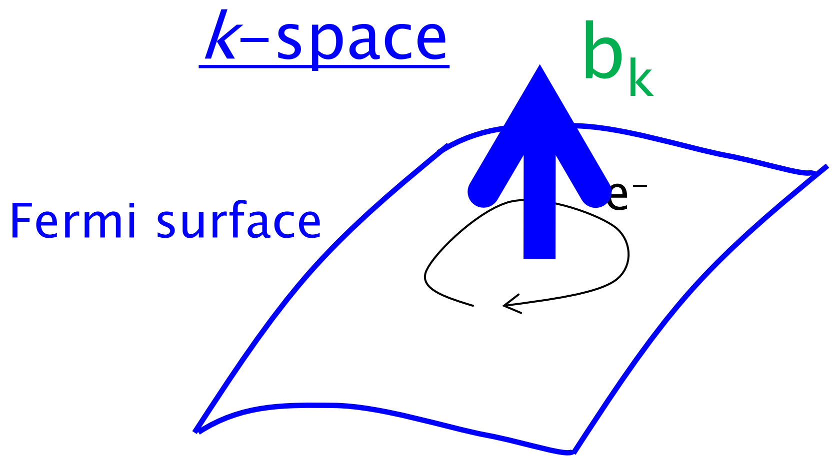
scalar spin chirality

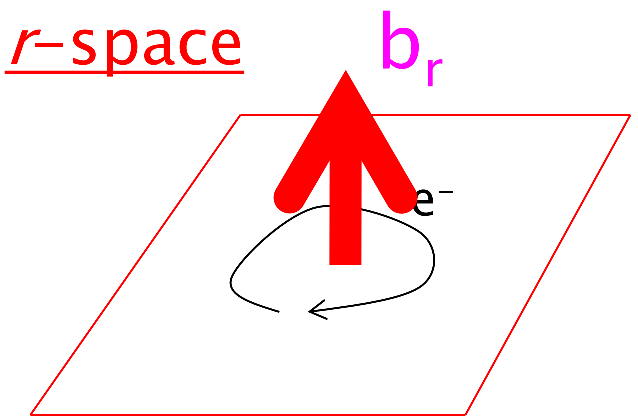


Equation of motion

$$\frac{d\boldsymbol{x}}{dt} = \frac{\partial \varepsilon}{\partial \boldsymbol{k}} + \frac{dk}{dt} \times b_{\mathrm{k}}$$

$$\frac{d\boldsymbol{k}}{dt} = -e\left(-\frac{\partial \varphi}{\partial \boldsymbol{r}} + \frac{dr}{dt} \times b_r \right)$$





B_{k} induced AHE

$$\sigma_{xy} \propto \tau^0, \rho_{xy} \propto \tau^{-2}$$

"dissipationless" nature

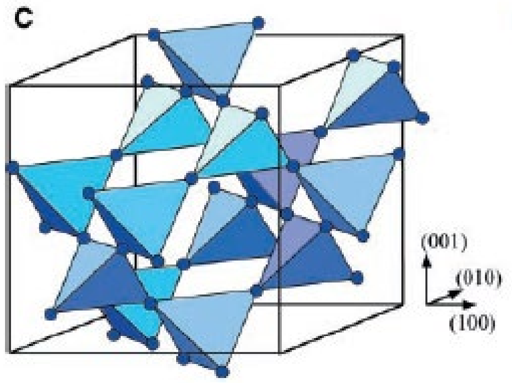
B_{r} induced AHE

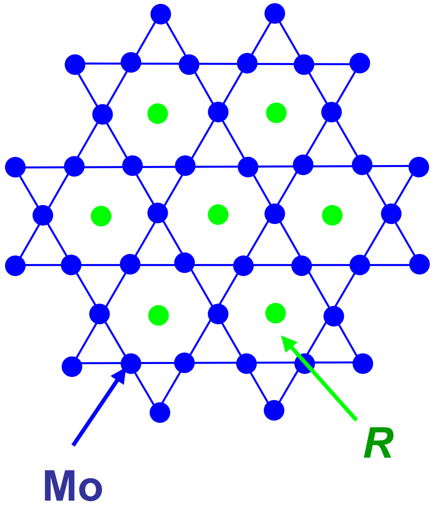
$$\sigma_{xy} \propto \tau^2, \rho_{xy} \propto \tau^0$$

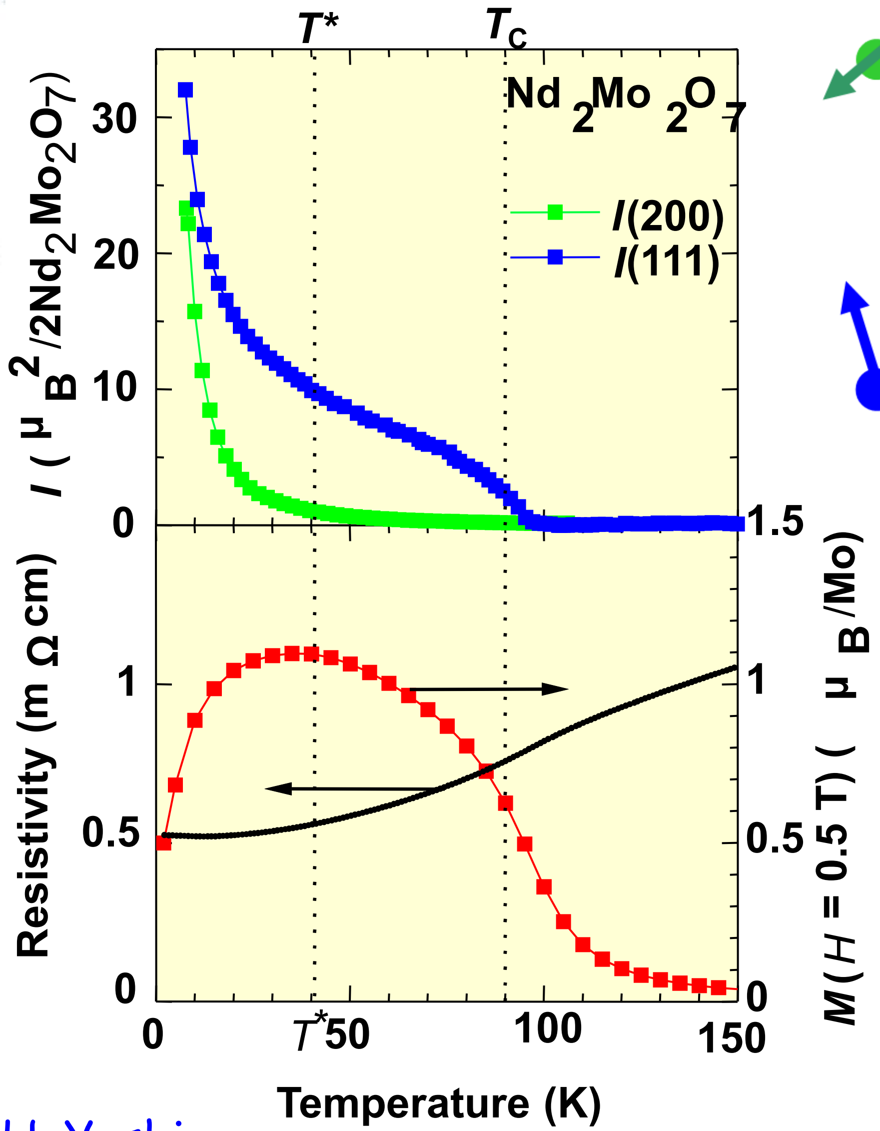
Cf. normal HE

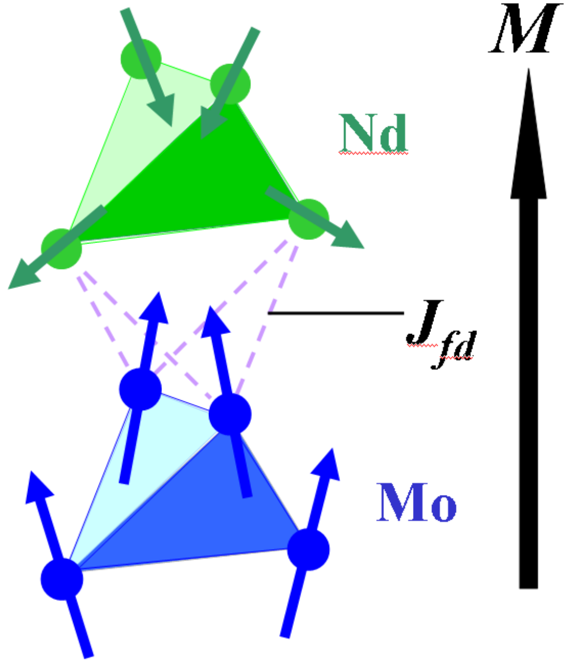
$$\rho_{xy} = B / ne, \sigma_{xy} \approx \sigma_{xx}{}^2 B / ne$$

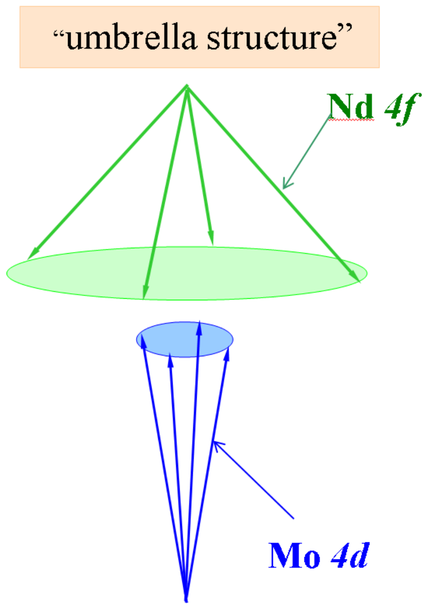
Pyrochlore $Nd_2Mo_2O_7$







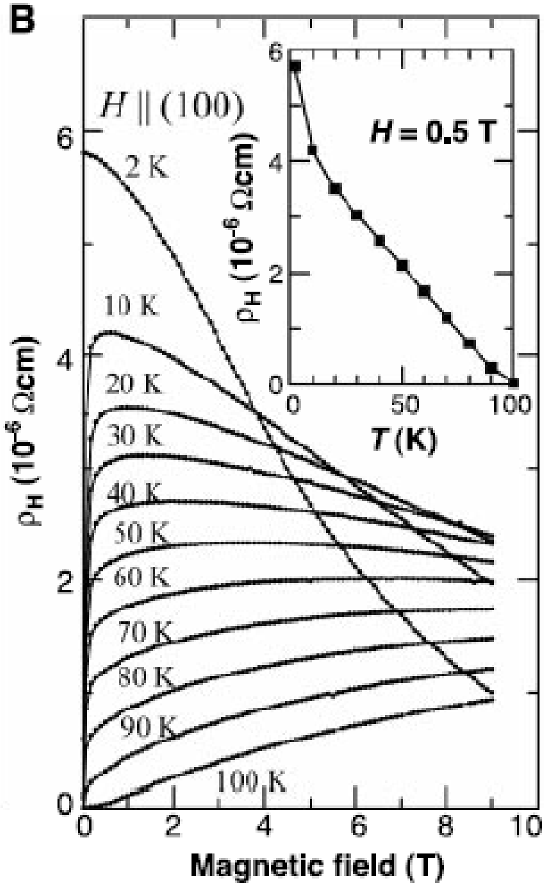


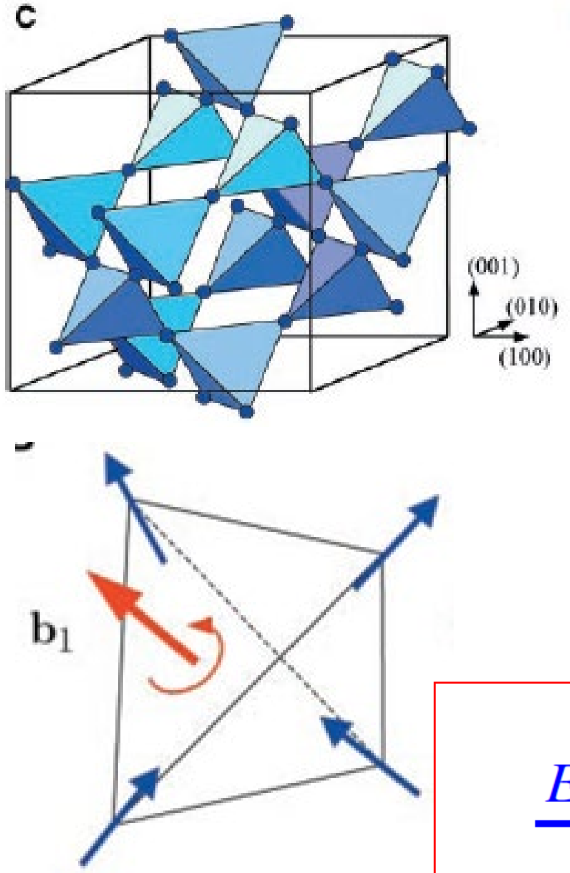


Y. Taguchi, Y. Oohara, H. Yoshizawa,
N. N. and Y. Tokura., Science 2001

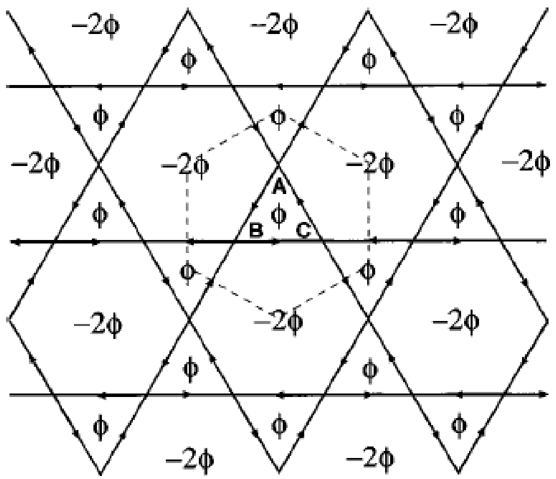
Spin chirality induced AHE in $Nd_2Mo_2O_7$

Y. Taguchi *et al.* Science 2001

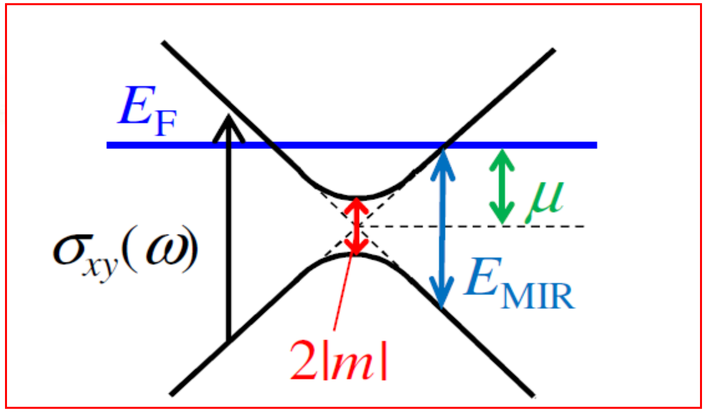




Total magnetic flux=0



Because of multiple loop



Iguchi et al. PRL (2009)

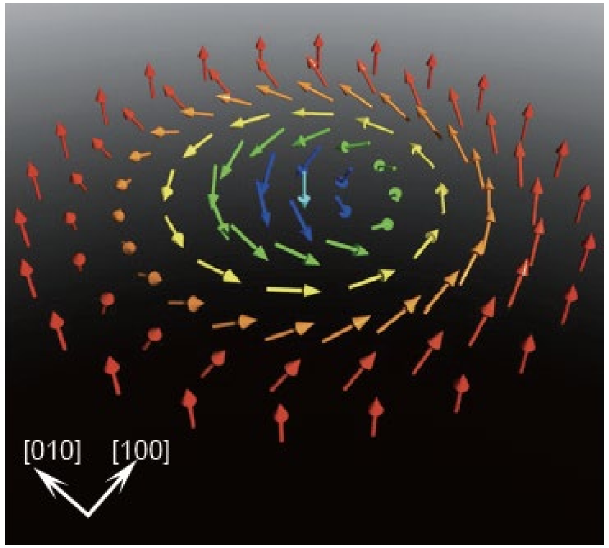
one flux quantum/(nm)^2~4000T !

→ Berry phase in momentum space
band crossing as monopole

Skyrmion as a topologically protected particle in magnets

a model for hardon T.H.R. Skyrme 1962

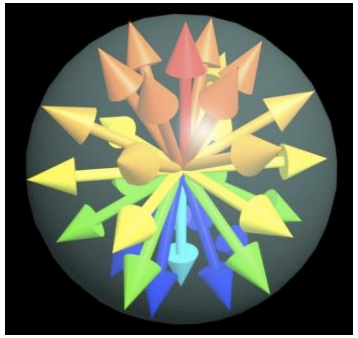
Skyrmion spin configuration



$$Q = \frac{1}{4\pi}\int dr^2 \vec{n}\cdot\left(\frac{\partial\vec{n}}{\partial x}\times\frac{\partial\vec{n}}{\partial y}\right)$$

Mapping to
a sphere

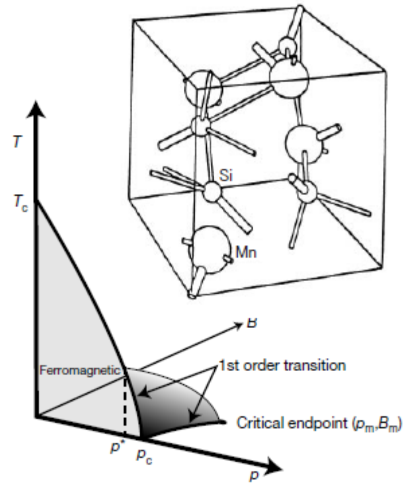
$$\pi_2(S^2) = Z$$



The integral of the solid angle
= topological number
How many times the mapping
wrap the unit sphere

Topological stability of skyrmion
as long as spins vary slowly

Skyrmions in MnSi



Pfleiderer, Rosch, Lonzarich, Loehneyzen et al

B20 structure without I-symmetry → DM int.

$$H_S = \int d^3x \left[\frac{J}{2a} (\nabla \mathbf{n})^2 + \frac{D}{a^2} \mathbf{n} \cdot [\nabla \times \mathbf{n}] - \frac{\mu}{a^3} \mathbf{H} \cdot \mathbf{n} \right]$$

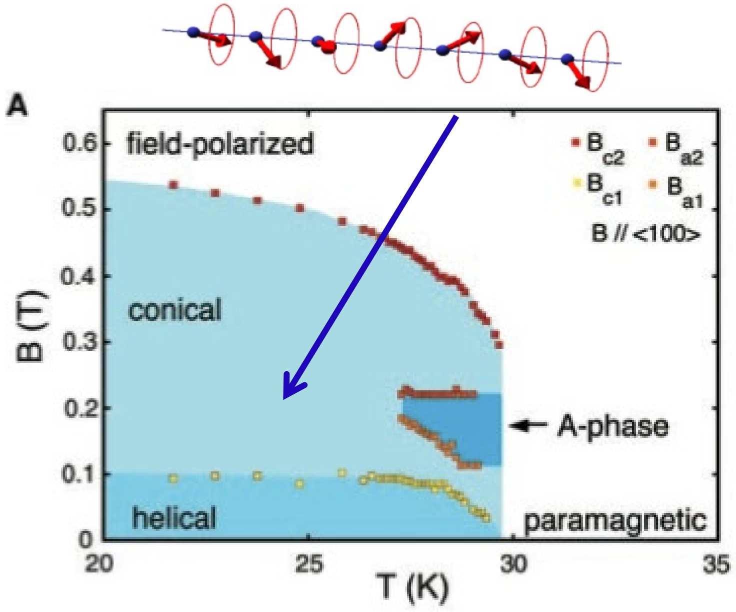
DM interaction → twist

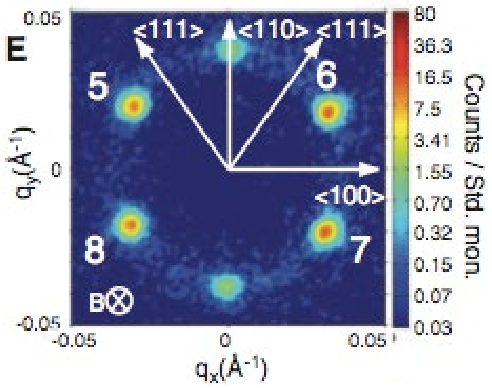
Theoretical works:

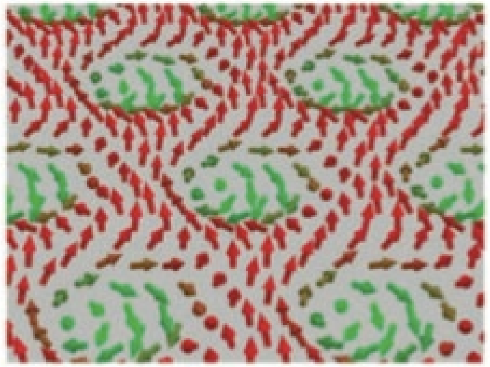
Bogdanov-Yablonskii 1989
Roessler-Bogdanov 2006
Binz-Vishwanath-Aji 2006

$$\xi = Ja / D$$

$$\sim (3 - 100)\text{nm} >> a$$

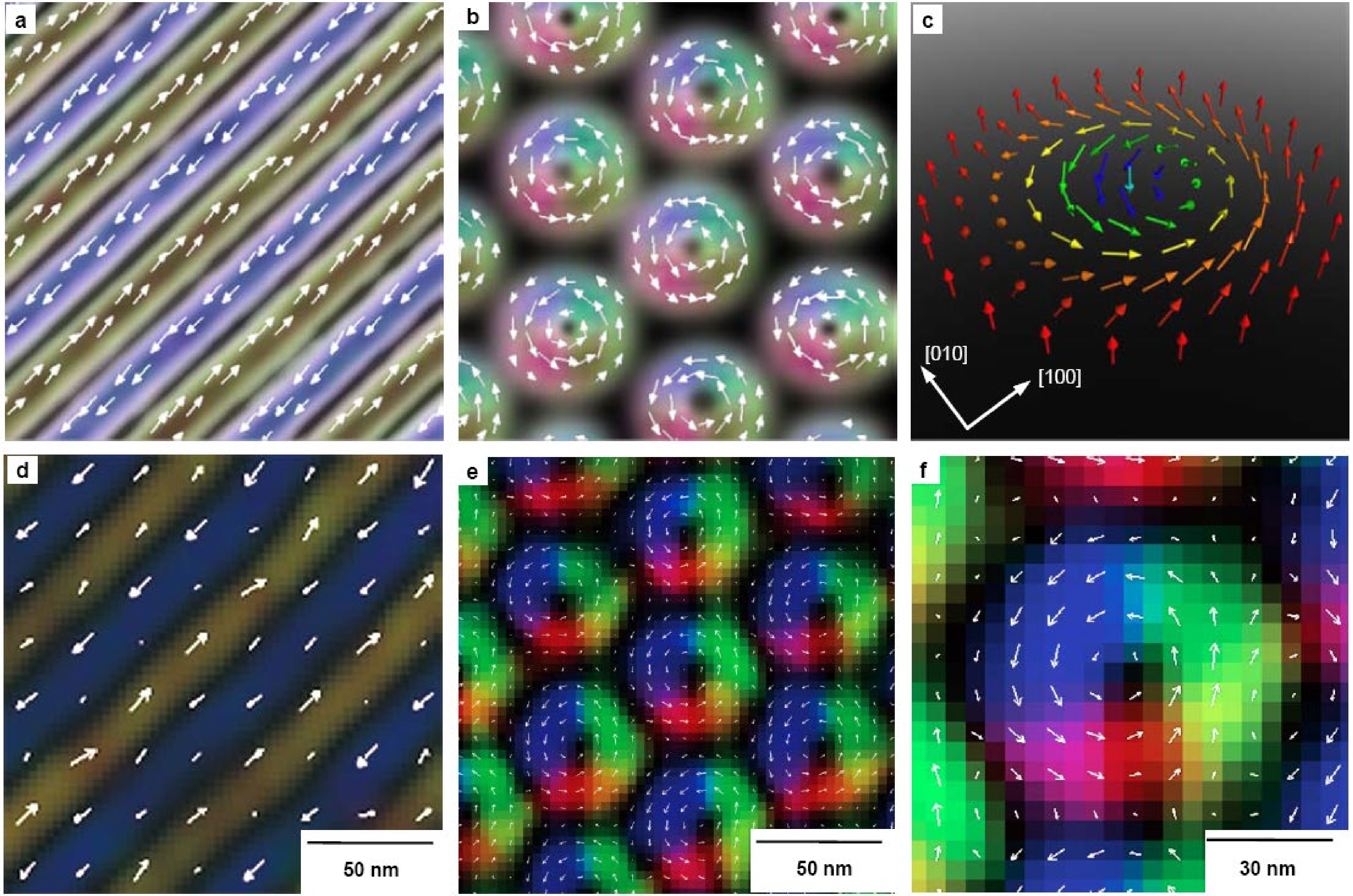




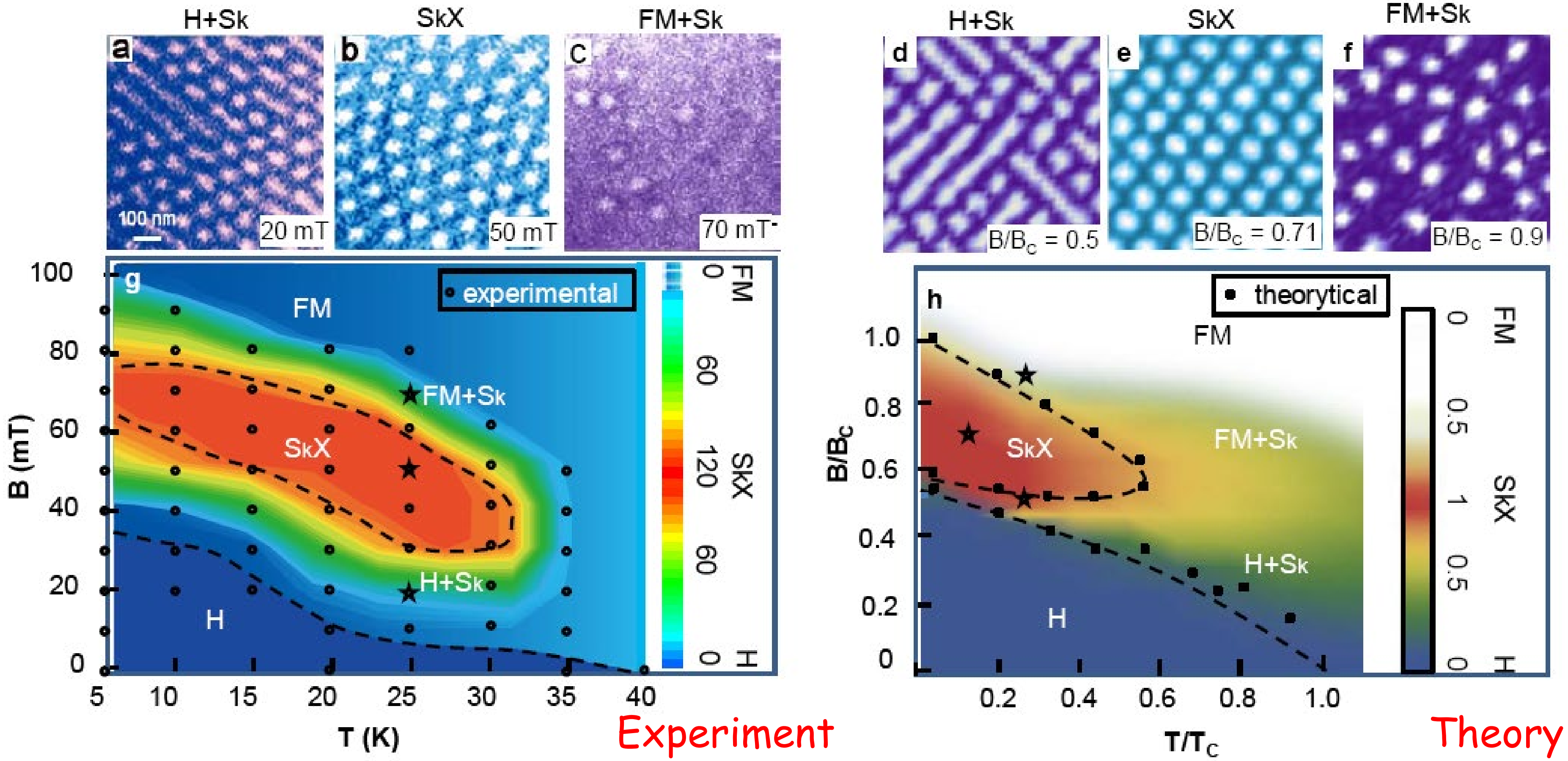


S. Mühlbauer *et. al.*,
Science 323 915 (2009)

Lorentz TEM observation of Skyrmion crystal in (Fe,Co)Si



X. Z. Yu, Y. Onose, N. Kanazawa, J. H. Park, J. H. Han, Y. Matsui, N. N. Y. Tokura
Nature (2010)



Energy density $\approx D^2/J = (D/J)^2 J \sim 10^{-4} J$

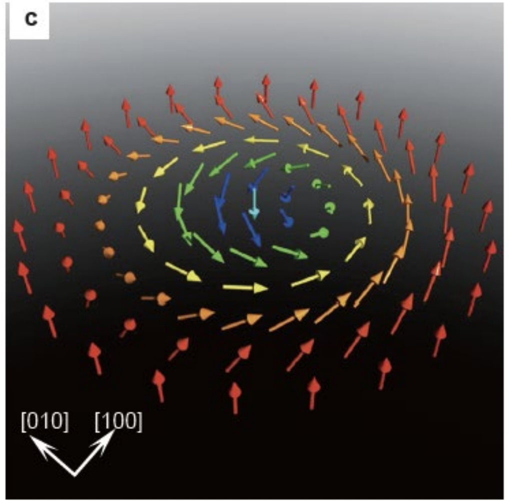
Size $\approx Ja/D >> a$ ⇨ Slowly varying continuum approximation

⇨ Topological stability protected by Sk number

Energy cost due to discontinuous spin cahnge $\approx J$

Coupled dynamics of conduction electrons and Skyrmion

J.D.Zang, J.H. Han, M.Mostovoy, and N.N.



Effective EMF due to spin texture acting on conduction electrons

$$\begin{cases} e_i = -\partial_i a_0 - \frac{1}{c}\dot{a}_i = \frac{\hbar}{2e}\left(\mathbf{n}\cdot\partial_i\mathbf{n}\times\dot{\mathbf{n}}\right), \\ h_i = [\boldsymbol{\nabla}\times\mathbf{a}]_i = \frac{\hbar c}{2e}\delta_{iz}\left(\mathbf{n}\cdot\partial_x\mathbf{n}\times\partial_y\mathbf{n}\right), \end{cases}$$

$$H_{\text{int}} = -\frac{1}{c}\int d^3x\,\mathbf{j}\cdot\mathbf{a}$$ Coupling term

Lorentz force

$$\frac{\partial n}{\partial t} + \mathbf{v}\cdot\frac{\partial n}{\partial \mathbf{x}} - e\left(\mathbf{E} + \mathbf{e} + \frac{1}{c}[\mathbf{v}\times(\mathbf{H}+\mathbf{h})]\right)\cdot\frac{\partial n}{\partial \mathbf{P}} = -\frac{\delta n}{\tau},$$

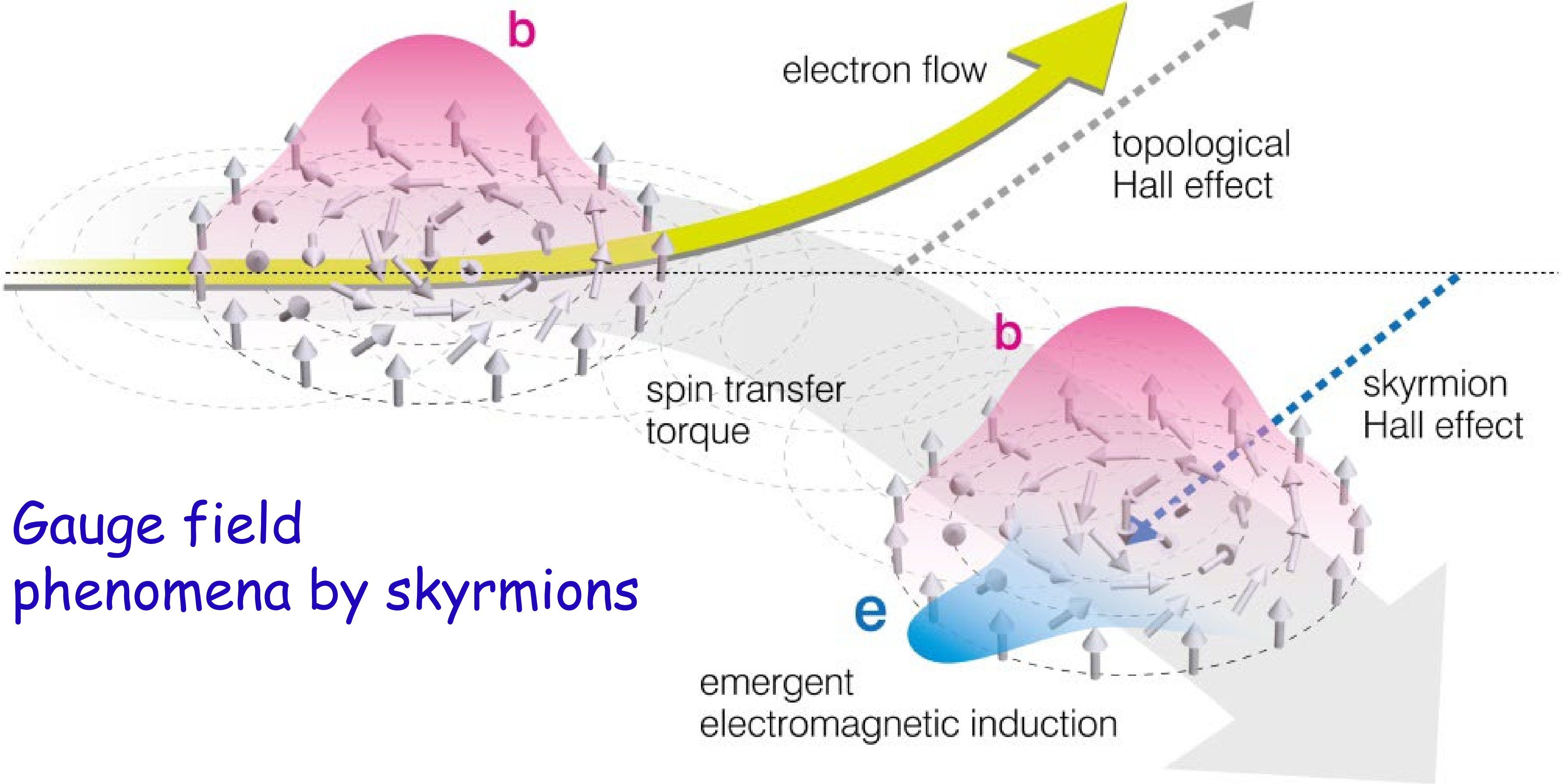
Boltzmann equation

$$\dot{\mathbf{n}} = \frac{\hbar\gamma}{2e}(\mathbf{j}\cdot\boldsymbol{\nabla})\mathbf{n} - \gamma\left[\mathbf{n}\times\frac{\delta H_S}{\delta\mathbf{n}}\right] + \alpha\left[\dot{\mathbf{n}}\times\mathbf{n}\right]$$

Spin transfer torque

LLG equation

Emergent electromagnetism



Gauge field
phenomena by skyrmions

Also multiferroic skyrmions in insulating magnets S. Seki

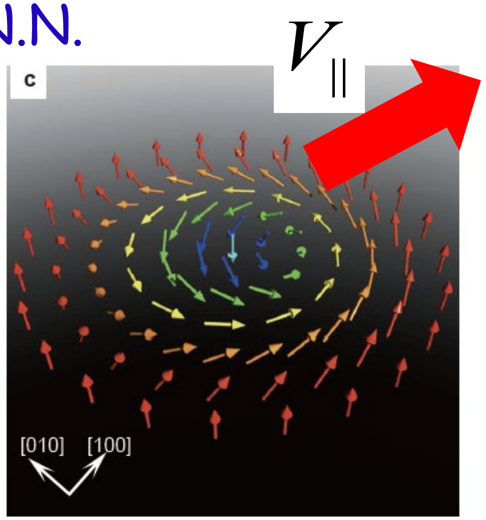
"Electromagnetic induction"

J.D.Zang, J.H.Han, M. Mostovoy, and N.N.

Moving magnetic flux produces
the transverse electric field

$$\mathbf{e} = -\frac{1}{c}\left[\mathbf{V}_{\parallel} \times \mathbf{h}\right]$$

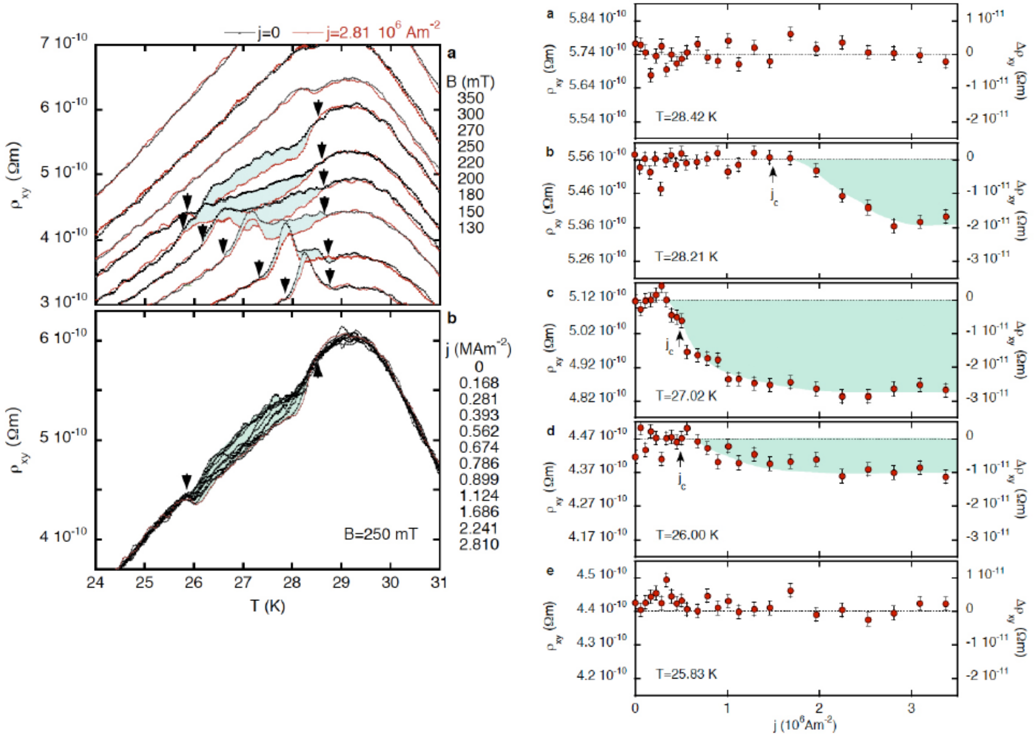
$$\Rightarrow \quad \frac{\Delta\sigma_{xy}}{\sigma} \approx -\frac{x}{2S+x}\,\frac{e\langle h_z\rangle\tau}{mc}$$

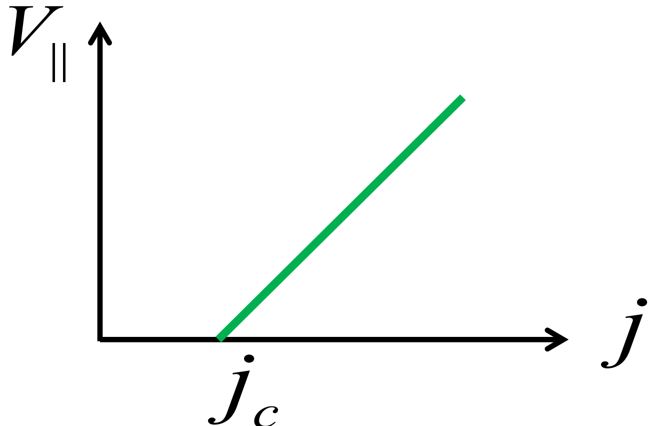


x Conduction electron number per site

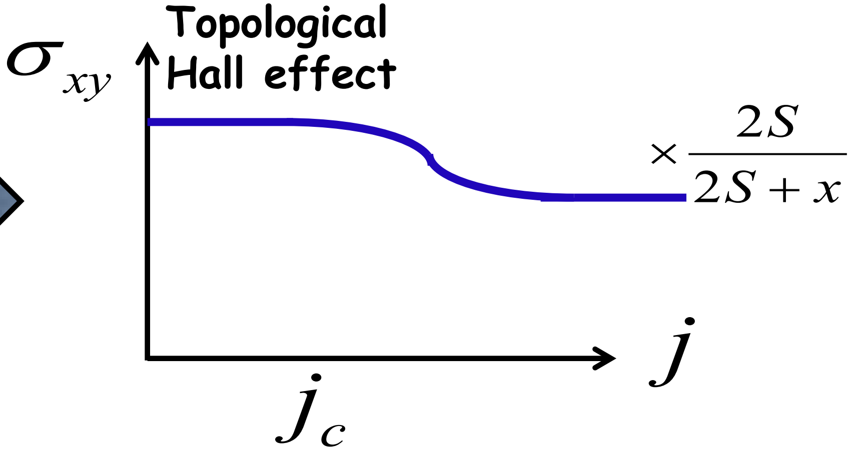
S Spin quantum number

c.f. $\dfrac{\sigma_{xy}^{top}}{\sigma} = \dfrac{e < h_z > \tau}{mc}$





$\Rightarrow$



C. Pfleiderer,
A.Rosch et al.
Nature Phys. ('12)

Wannier function and its localization

N. Marzari and D. Vanderbilt, PRB56, 12847 (1997)

$$|w_{nR}\rangle = \frac{1}{\sqrt{N}}\sum_k e^{-ik\cdot R}|\psi_{nk}\rangle \qquad |\psi_{nk}\rangle = \frac{1}{\sqrt{N}}\sum_R e^{ik\cdot R}|w_{nR}\rangle$$

Wannier function — Bloch function

$$\langle w_{nR}|w_{mR'}\rangle = \frac{1}{N}\sum_{k,k'} e^{ik\cdot R}e^{-ik'\cdot R'}\langle\psi_{nk}|\psi_{mk'}\rangle = \frac{1}{N}\sum_{k,k'}\delta_{n,m}\delta_{k,k'}e^{ik\cdot(R-R')} = \delta_{R,R'}\delta_{n,m}$$

Localization length of Wannier function

$$\Sigma_n = \langle w_{n0}|\Delta\hat{x}^2|w_{n0}\rangle = \langle w_{n0}|\hat{x}^2|w_{n0}\rangle - \langle w_{n0}|\hat{x}|w_{n0}\rangle^2$$

$$= \frac{1}{N}\sum_k\sum_{i=1}^{d}[g_{n,ii}(k) + (A^i_{nk} - \bar{A}^i{}_n)^2]$$

Integral of quantum metric over k giving the lower bound

Berry connection which depends on the gauge choice

Superfluidity of flat-band superconductor

Torma group, Bernevig group, S.A.Chen, K.T.Law, Phys. Rev. Lett. **132**, 026002 (2024).

$$H_{\text{int.}} = -g \int dx \psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x)\psi_\downarrow(x)\psi_\uparrow(x) \qquad \psi_\sigma(x) = \frac{1}{\sqrt{V}} \sum_{n,k} \phi_{nk}(x) c_{nk\sigma}$$

$$H_{\text{mean int.}} = -g\Delta_0 \int dx e^{iq\cdot x} \psi_\uparrow^\dagger(x)\psi_\downarrow^\dagger(x) + \text{h.c.}$$

$$= -g\Delta_0 \sum_{n,k} c^\dagger_{n-k+q/2} c^\dagger_{nk+q/2} \langle u_{nk+q/2} | u^*_{n-k+q/2}\rangle + \text{h.c.}$$

$$|u^*_{n-k+q/2}\rangle = |u_{nk-q/2}\rangle \qquad \Gamma(k,q) = \langle u_{nk+q/2} | u_{nk-q/2}\rangle. \qquad |\Gamma(k,q)|^2 = 1 - \sum_{a,b} g_{ab}(k) q_a q_b$$

Time-reversal symmetry

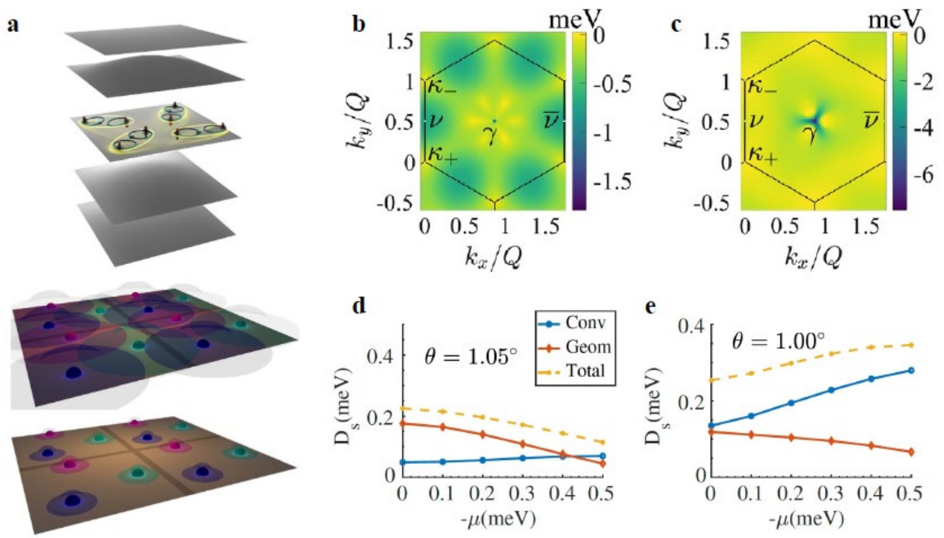
$$F_2 = \sum_q (g - g^2\chi(q))\Delta(q)\Delta(-q). \qquad \chi(q) = \frac{1}{\beta}\sum_{\omega_n}\frac{1}{N}\sum_k |\Gamma(k,q)|^2 G(k+q/2)G(-k+q/2)$$

$$\chi_2(q) = \sum_{a,b} q_a q_b \frac{1}{4N} \sum_k \frac{\tanh(\beta\varepsilon_n(k)/2)}{\varepsilon_n(k)} g_{ab}(k) \qquad \rho_s = \rho_s^{\text{kinetic}} + \rho_s^{\text{geometric}}$$

Geometric contribution

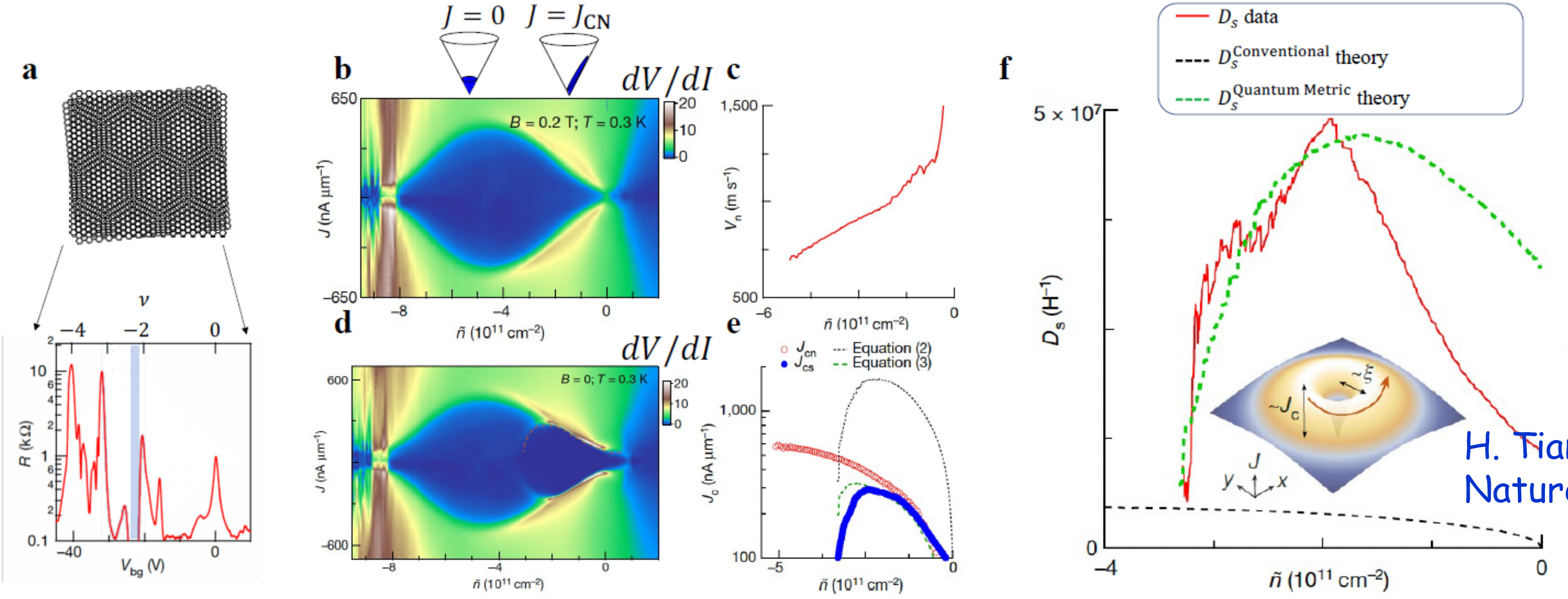
$$\xi = \sqrt{\xi^2_{\text{BCS}} + \ell^2_{\text{qm}}}\,.$$

Twisted Bilayer Graphene theory and experiment



Theory

S. Peotta, P. Torma (2015), Nature Commun. 6, 8944 (2015).
X. Hu et al., PRL 123, 237002 (2019)

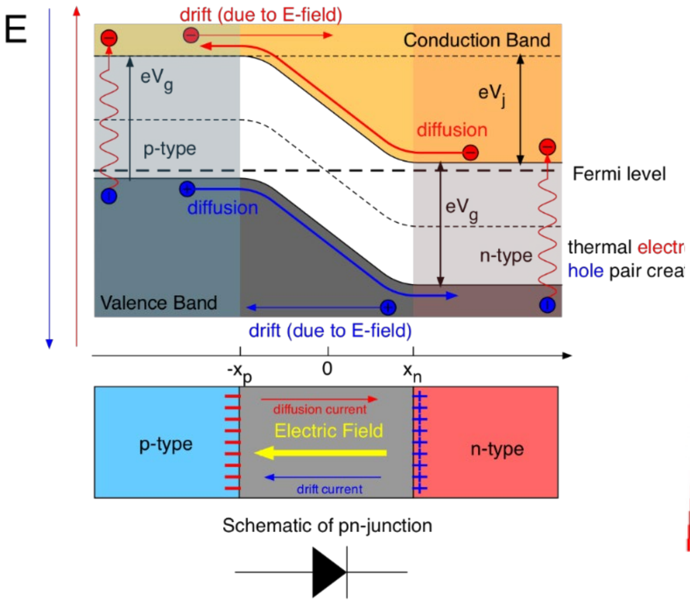


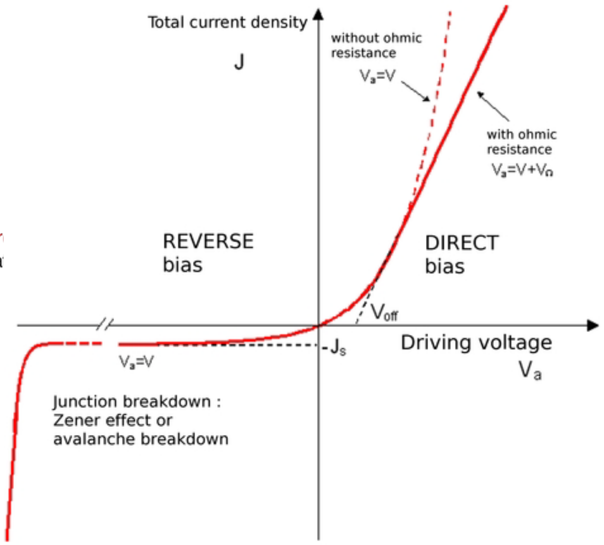
Experiment
H. Tian et al.,
Nature 614 (7948), 440–444 (2023)

Nonlinear responses

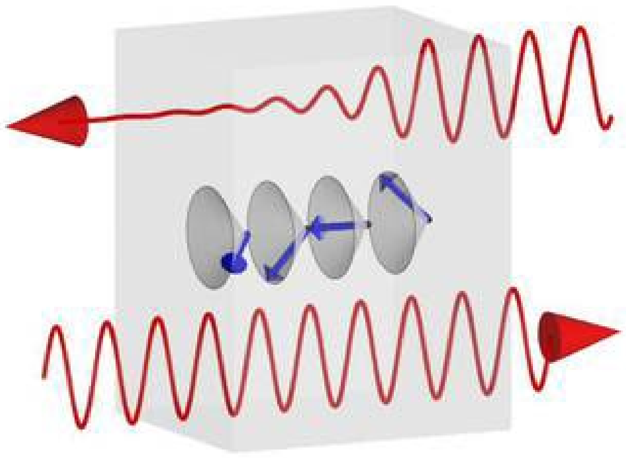
Directional response is useful

pn junction





optical isolator



http://quantumwise.com/publications/tutorials/item/828-silicon-p-n-junction

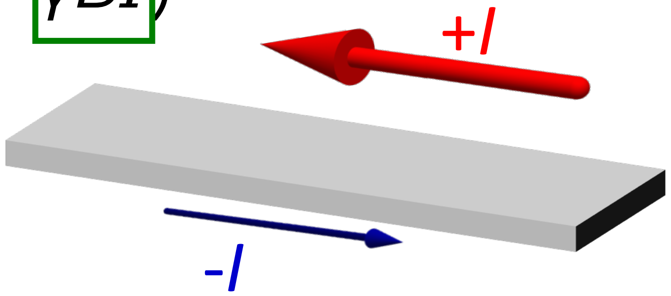
http://www.optique-ingenieur.org/en/courses/OPI_ang_M05_C02/co/Contenu_09.html

Non-reciprocal transport in non-centrosymmetric system

Time-reversal symmetry of microscopic dynamics vs irreversibility

$$\sigma_{\mu\nu}(k,\omega,B)=\sigma_{\nu\mu}(-k,\omega,-B)$$ Onsager's reciprocal relation for linear response

$k \rightarrow I$ G. L. J. A. Rikken (2001) $R = R_0\,(1 + \beta B^2 + \boxed{\gamma BI})$



Nonreciprocal Response	Linear Response	Nonlinear Response
Time-reversal Unbroken	Forbidden	Shift current Nonlinear Hall effect pn junction
Time-reversal Broken	Optical ME effect Magnetochiral effect Nonreciprocal magnon	Nonreciprocal nonlinear optical effect Electric magnetochiral effect Inverse Edelstein effect Magnetochiral anisotropy

Berry curvature

Time-reversal $b(k)=-b(-k)$

Inversion $b(k)=b(-k)$

$\Rightarrow$ $b(k)=0$

Y. Tokura and N. Nagaosa, Nature Commun. 2018

Second order nonlinear and nonreciprocal transport

Y. Gao, S.S. Yang, and Q.Niu, PRL112, 166601 (2014)

$$J = -e\dot{x}_c = -e\frac{\partial \varepsilon_n(k)}{\partial k} - e^2 E \times b(k)$$

$$J_i = \sigma^{(2)}_{ijl} E_j E_l$$

$$\sigma^{(2)}_{ijl} = \int (dk) e^3 \tau^2 \frac{\partial v_{k,i}}{\partial k_j} v_{k,l} \frac{\partial f_{\mathrm{eq.}}}{\partial \varepsilon}$$

Boltzmann transport both T and P broken

$$(J_{\mathrm{NLH}})_i = \int (dk) \left(-e\tau E_j \cdot \frac{\partial f(k)}{\partial k_j} \right) (-e^2 \epsilon_{ilm} E_l b_m(k))$$

$$= -e^3 \tau \int (dk) \epsilon_{ilm} f(k) \frac{\partial b_m(k)}{\partial k_j} E_j E_l \qquad (282)$$

Nonlinear Hall effect
Berry curvature dipole

$$a^1_{na} = -e \sum_{m(\neq n)} 2\mathrm{Re}\left(\frac{(v_a)_{nm}(v_b)_{mn}}{\varepsilon^3_{nm}} \right) E_b = -eG^n_{ab} E_b \quad G_{ab} = -\frac{1}{4h(k)} \partial_a \vec{n}(k) \cdot \partial_b \vec{n}(k) = -\frac{1}{h(k)} g_{ab}(k)$$

Correction to Berry connection by E

$$J^1_i = e^3 \int (dk) \epsilon_{ijl} (v_i(k) G_{jl}(k) - v_j(k) G_{il}(k)) \frac{\partial f(k)}{\partial \varepsilon} E_j E_l$$

Intrinsic nonreciprocal transport
due to quantum metric

PHYSICAL REVIEW B **106**, 125114 (2022)

Dissipation and geometry in nonlinear quantum transports of multiband electronic systems

Yoshihiro Michishita[1,*] and Naoto Nagaosa[1,2,†]

$$\sigma_{\rm dc}^{\alpha;\beta\gamma} = \sigma_{\rm M}^{\alpha;\beta\gamma} + \sigma_{\rm G}^{\alpha;\beta\gamma}$$

$$\sigma_{\rm M}^{\alpha;\beta\gamma} = \sum_{nml}\sum_{\omega_M>0} {\rm Re}\Bigg[\frac{\partial}{\partial\omega}\Bigg\{2\mathcal{J}^{\alpha}_{nm}\frac{\partial G^R_m}{\partial\omega}\mathcal{J}^{\beta}_{ml}G^R_l\mathcal{J}^{\gamma}_{ln}(G^R_n-G^A_n) + \mathcal{J}^{\alpha}_{nm}\frac{\partial G^R_m}{\partial\omega}\mathcal{J}^{\beta\gamma}_{mn}(G^R_n-G^A_n)\Bigg\} + \{\beta\leftrightarrow\gamma\}\Bigg]_{\omega=i\omega_M},$$

Contribution
From Matsubara poles

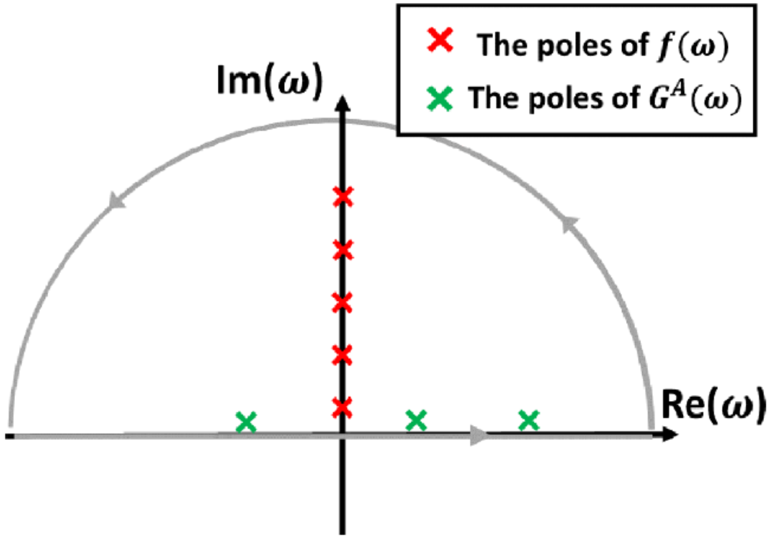


FIG. 1. Path integration in the dc limit.

$$\sigma_{\rm G}^{\alpha;\beta\gamma} = \sigma_{\rm Drude}^{\alpha;\beta\gamma} + \sigma_{\rm BCD}^{\alpha;\beta\gamma} + \sigma_{\rm ChS}^{\alpha;\beta\gamma} + \sigma_{\rm gBC}^{\alpha;\beta\gamma} + O(\tau^{-2})$$

$$\sigma_{\rm Drude}^{\alpha;\beta\gamma} \simeq 2\sum_{k}\sum_{n}\tau^2\mathcal{J}^{\alpha}_{n}\partial^{\beta}\partial^{\gamma}f(\epsilon_n).$$

Drude and Geometric contributions

Usually the contour integral is reduced to that along the real axis.

See also D. Kaplan, T. Holder, and B. Yan, 2022

Geometric terms in nonreciprocal conductivity

$$\sigma_{\rm G}^{\alpha;\beta\gamma} = \sigma_{\rm Drude}^{\alpha;\beta\gamma} + \sigma_{\rm BCD}^{\alpha;\beta\gamma} + \sigma_{\rm ChS}^{\alpha;\beta\gamma} + \sigma_{\rm gBC}^{\alpha;\beta\gamma} + O(\tau^{-2})$$

Berry curvature dipole term

$$\sigma_{\rm BCD}^{\alpha;\beta\gamma} = \sigma_{\rm BCD:re}^{\alpha;\beta\gamma} + \sigma_{\rm BCD:im}^{\alpha;\beta\gamma},$$

$$\sigma_{\rm BCD:re}^{\alpha;\beta\gamma} \simeq 2\tau \sum_{\boldsymbol{k}} \sum_{nm} \partial^{\gamma}(\Omega_{S;n,m}^{\alpha\beta}) f(\epsilon_n),$$

$$\sigma_{\rm BCD:im}^{\alpha;\beta\gamma} \simeq \sum_{\boldsymbol{k}} \sum_{nm} \frac{\Omega_{S;n}^{\alpha\beta}\mathcal{J}_n^{\gamma}}{\epsilon_{nm}\tau} \left(\frac{\partial^2 f}{\partial\omega^2}\right)_{\epsilon_n},$$

$$\Omega_{S;n,m}^{\alpha\beta} = \frac{-i(\mathcal{J}_{nm}^{\alpha}\mathcal{J}_{mn}^{\beta} - \mathcal{J}_{nm}^{\beta}\mathcal{J}_{mn}^{\alpha})}{\epsilon_{nm}^2 + 4\eta^2}.$$

Chirstoffel symbol term

$$\sigma_{\rm ChS}^{\alpha;\beta\gamma} = \sigma_{\rm ChS:I}^{\alpha;\beta\gamma} + \sigma_{\rm ChS:II}^{\alpha;\beta\gamma},$$

$$\sigma_{\rm ChS:I}^{\alpha;\beta\gamma} = 2 \sum_{\boldsymbol{k}} \sum_{n} \Gamma_{S;n}^{\alpha;\beta\gamma} \left(-\frac{\partial f}{\partial\omega}\right)_{\epsilon_n},$$

$$\sigma_{\rm ChS:II}^{\alpha;\beta\gamma} \simeq 2 \sum_{\boldsymbol{k}} \sum_{n} \Gamma_{S';n}^{\alpha;\beta\gamma} \left(-\frac{\partial f}{\partial\omega}\right)_{\epsilon_n},$$

$$\Gamma_n^{\alpha;\beta\gamma} = \frac{1}{2}(\partial^{\gamma} g_n^{\alpha\beta} + \partial^{\beta} g_n^{\gamma\alpha} - \partial^{\alpha} g_n^{\beta\gamma}) = \frac{1}{2}(\langle\partial^{\alpha} n|\partial^{\beta\gamma} n\rangle + \langle\partial^{\beta\gamma} n|\partial^{\alpha} n\rangle),$$

Generalized Berry curvature term

$$\sigma_{\rm gBC}^{\alpha;\beta\gamma} = \sigma_{\rm gBC:re}^{\alpha;\beta\gamma} + \sigma_{\rm gBC:im}^{\alpha;\beta\gamma} + \sigma_{\rm gBC:add}^{\alpha;\beta\gamma},$$

$$\sigma_{\rm gBC:re}^{\alpha;\beta\gamma} \simeq \sum_{\boldsymbol{k}} \sum_{n,m(\neq n)} \frac{\Omega_{S';n,m}^{\alpha,\beta\gamma}}{\epsilon_{nm}\tau} \left(-\frac{\partial f}{\partial\omega}\right)_{\epsilon_n},$$

$$\sigma_{\rm gBC:add}^{\alpha;\beta\gamma} \simeq 2 \sum_{\boldsymbol{k}} \sum_{n,m,l(\neq n)} \left\{ \frac{{\rm Im}(\mathcal{J}_{nm}^{\alpha}\mathcal{J}_{ml}^{\beta}\mathcal{J}_{ln}^{\gamma})}{\epsilon_{nm}^2\epsilon_{nl}} \times \left(\frac{1}{\epsilon_{nm}\tau} - \frac{1}{\epsilon_{nl}\tau}\right)\left(-\frac{\partial f}{\partial\omega}\right)_{\epsilon_n} + (\beta\leftrightarrow\gamma)\right\},$$

$$\sigma_{\rm gBC:im}^{\alpha;\beta\gamma} \simeq -\sum_{\boldsymbol{k}} \sum_{n} \frac{\Omega_{S';n}^{\alpha,\beta\gamma}}{\tau} \left(-\frac{\partial^2 f}{\partial\omega^2}\right)_{\epsilon_n},$$

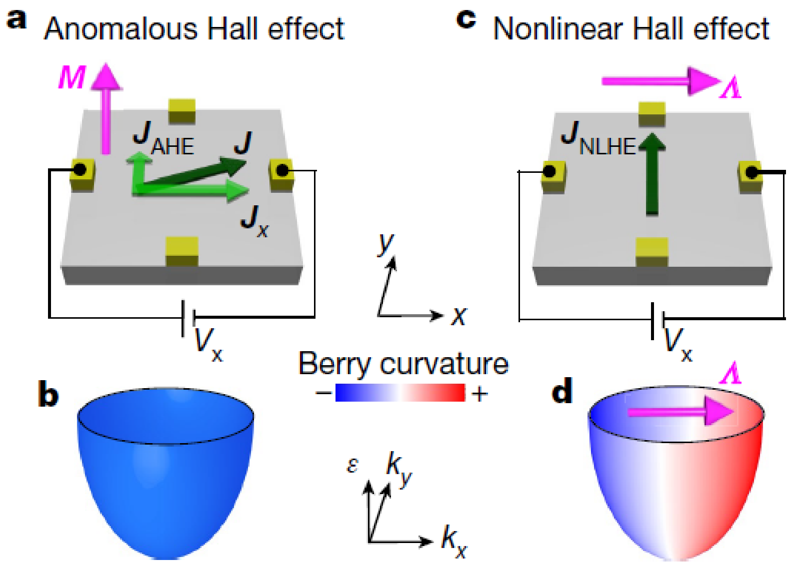
$$\Omega_{n,m}^{\alpha,\beta\gamma} = 2{\rm Im}[\langle\partial^{\alpha} n|m\rangle \langle m|\partial^{\beta\gamma} n\rangle]$$

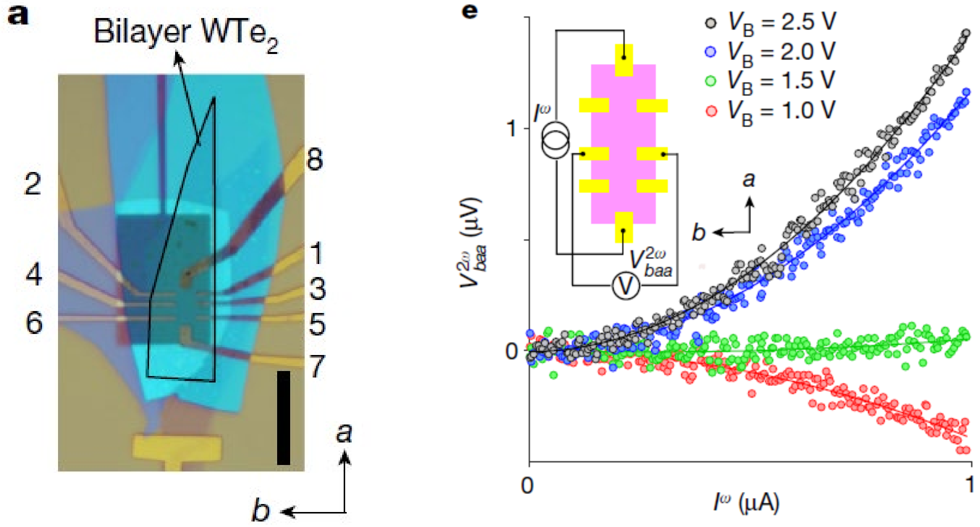
Nonlinear Hall effect

Moore-Orenstein PRL 2010
Sodemann-Fu PRL 2015
Q. Ma et al. Nature 2019

$$D_{ab} = \int_k f_0 \, (\partial_a \Omega_b)$$

dipole moment
of Berry curvature





Quantum-metric-induced nonlinear transport in a topological antiferromagnet

Nature | Vol 621 | 21 September 2023 | 487

https://doi.org/10.1038/s41586-023-06363-3

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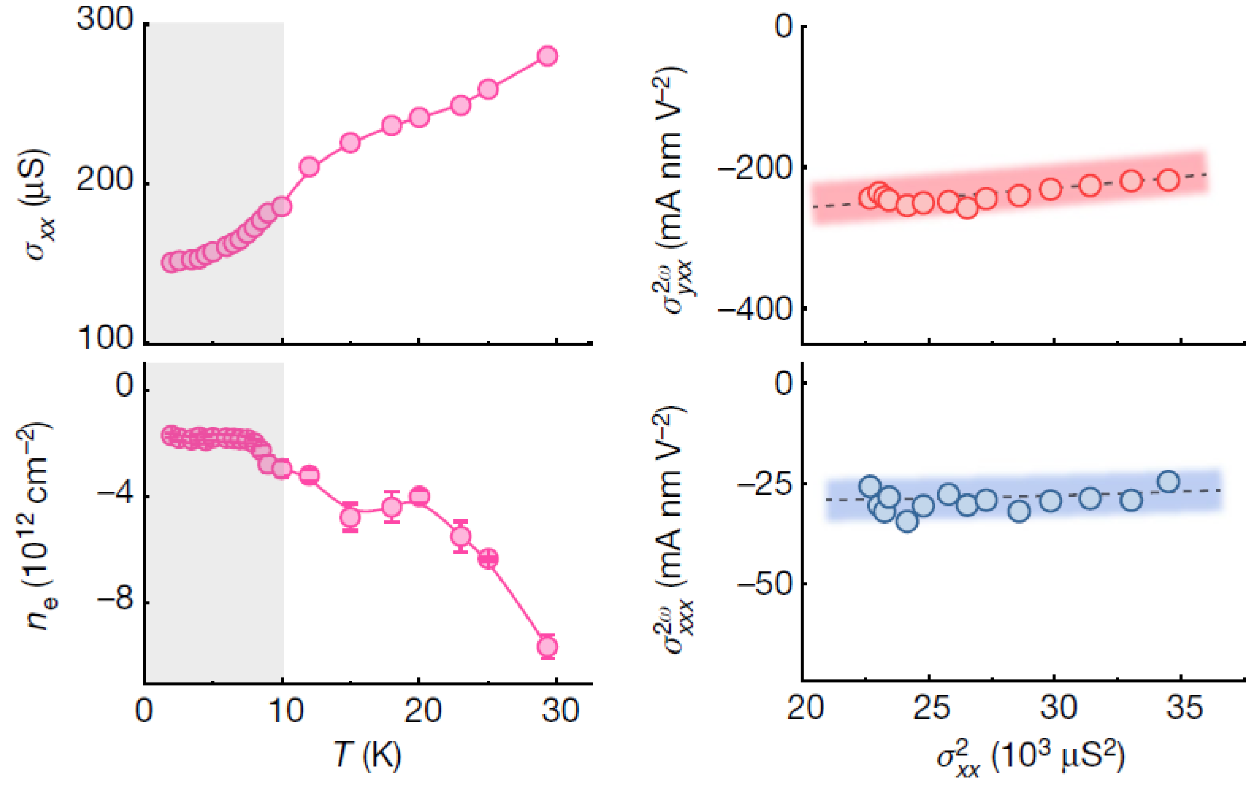
Naizhou Wang[1], Daniel Kaplan[2], Zhaowei Zhang[1], Tobias Holder[2], Ning Cao[3], Aifeng Wang[3], Xiaoyuan Zhou[3], Feifei Zhou[1], Zhengzhi Jiang[1], Chusheng Zhang[1], Shihao Ru[1], Hongbing Cai[1,4], Kenji Watanabe[5], Takashi Taniguchi[6], Binghai Yan[2]✉ & Weibo Gao[1,4,7]✉



MnBi2Te4

$$\sigma^0_{baa} = -\frac{e^3}{\hbar^2}\int d^2k f_n \left(2\partial_{k_b} G^{aa}_n - \partial_{k_a} G^{ab}_n\right)$$

$$\sigma^{2\omega} = \eta_2 (\sigma^{\omega}_{xx})^2 + \eta_0$$

Intrinsic term

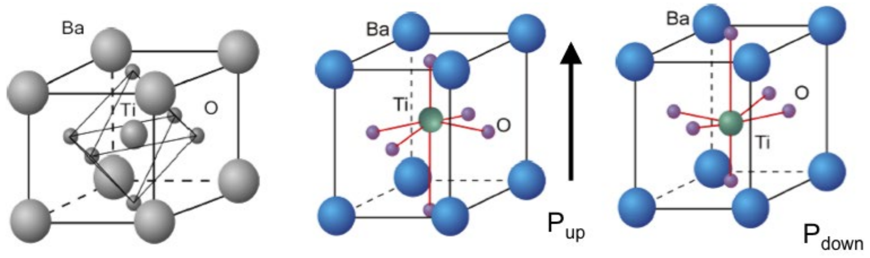
LETTER

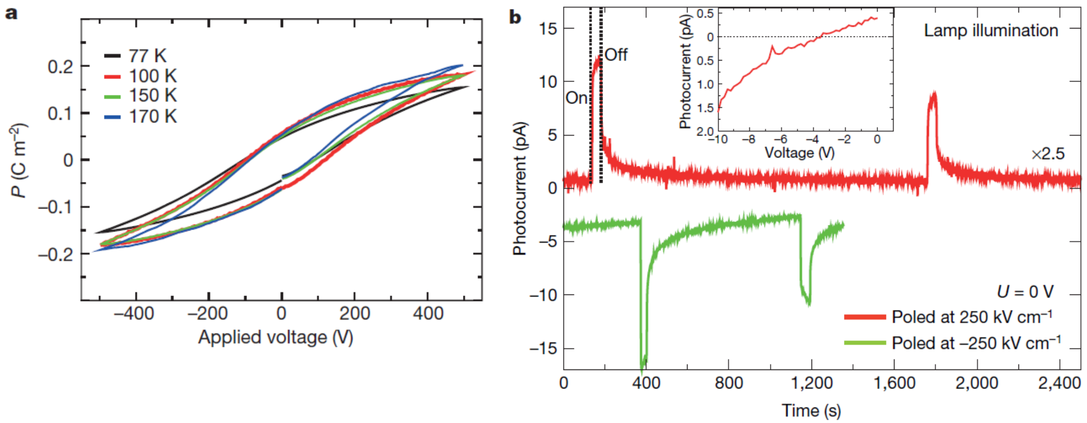
Nature 2013

doi:10.1038/nature12622

Perovskite oxides for visible-light-absorbing ferroelectric and photovoltaic materials

Ilya Grinberg[1], D. Vincent West[2], Maria Torres[3], Gaoyang Gou[1], David M. Stein[2], Liyan Wu[2], Guannan Chen[3], Eric M. Gallo[3], Andrew R. Akbashev[3], Peter K. Davies[2], Jonathan E. Spanier[3] & Andrew M. Rappe[1]





$[KNbO_3]_{1-x}[BaNi_{1/2}Nb_{1/2}O_{3-\delta}]_x$ (KBNNO)

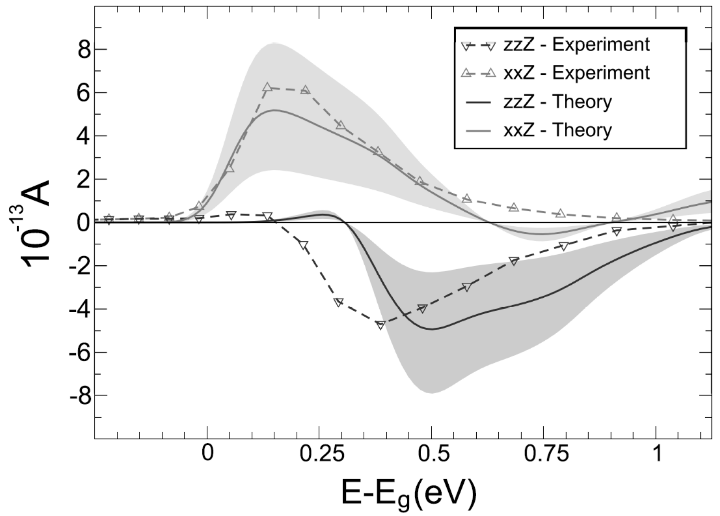
Bulk photovoltaic effect
in noncentrosymmetric crystal
→ Shift-current ?

Calculation of shift current

$$J_q = \sigma_q^{rs} E_r E_s$$

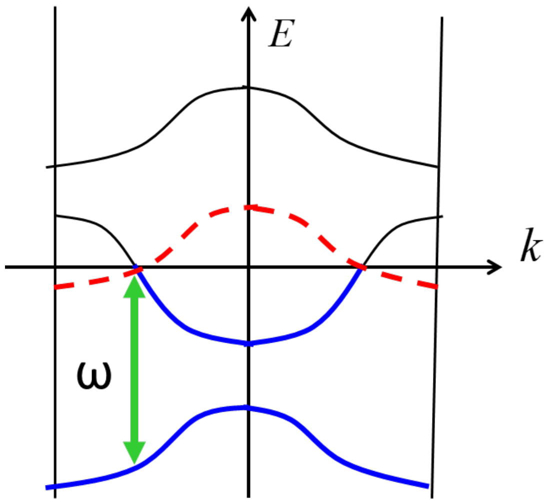
$$\sigma_q^{rs}(\omega) = \pi e \left(\frac{e}{m\hbar\omega}\right)^2 \sum_{n',n''} \int d\mathbf{k} \left(f[n''\mathbf{k}] - f[n'\mathbf{k}]\right)$$
$$\times \langle n'\mathbf{k}| \hat{P}_r |n''\mathbf{k}\rangle \langle n''\mathbf{k}| \hat{P}_s |n'\mathbf{k}\rangle$$
$$\times \left(-\frac{\partial \phi_{n'n''}(\mathbf{k},\mathbf{k})}{\partial k_q} - [\chi_{n''q}(\mathbf{k}) - \chi_{n'q}(\mathbf{k})]\right)$$
$$\times \delta\left(\omega_{n''}(\mathbf{k}) - \omega_{n'}(\mathbf{k}) \pm \omega\right)$$

R. von Baltz and W. Kraut, Physical Review B 23,5590 (1981).
J. Sipe and A. Shkrebtii, Physical Review B 61, 5337 (2000).
S. M. Young and A. M. Rappe, Phys. Rev. Lett. 109, 116601 (2012)



Geometric nature of Floquet bands

T.Morimoto and N.N. Science Adv. 2, E1501524 (2016)



Floquet bands

$$h = \begin{pmatrix} \epsilon_1^0 & iFv_{12}^0 \\ -iFv_{21}^0 & \epsilon_2^0 \end{pmatrix} \equiv \epsilon + \boldsymbol{d}\cdot\boldsymbol{\sigma}$$

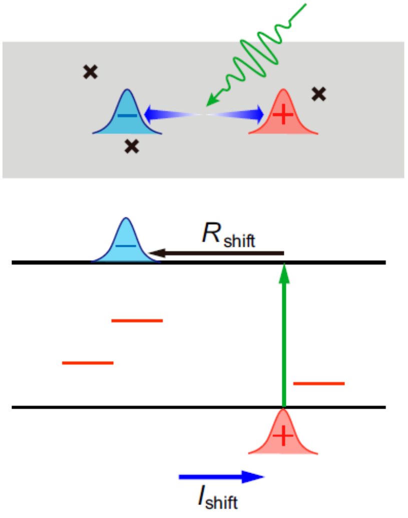
$$G^< = \frac{(\omega-\epsilon-i\Gamma+\boldsymbol{d}\cdot\boldsymbol{\sigma})\Sigma^<(\omega-\epsilon+i\Gamma+\boldsymbol{d}\cdot\boldsymbol{\sigma})}{[(\omega-\epsilon-i\Gamma)^2-d^2][(\omega-\epsilon+i\Gamma)^2-d^2]}$$

Conduction and valence bands are coupled to particle bath Independently

$$J = \frac{\Gamma(d_x b_y - d_y b_x)}{2(d^2+\Gamma^2)} + \frac{d_z^2 b_z}{4(d^2+\Gamma^2)} + \frac{b_z}{4} \equiv J_1 + J_2 + J_3.$$

$$b = \partial d / \partial k$$

Current



$$J_1 = F^2 \frac{\Gamma}{2(d_z^2+F^2|v_{12}^0|^2+\Gamma^2)}|v_{12}^0|^2 R_k. \quad R_k = \left[\frac{\partial}{\partial k}\mathrm{Im}(\log v_{21}^0) + a_{22} - a_{11}\right]$$

$$= \frac{\pi E^2}{2\Omega^2}\frac{\Gamma}{\sqrt{\frac{E^2}{\Omega^2}+\Gamma^2}}\delta(d_z)|v_{12}^0|^2 R_k.$$

Stimulated emission

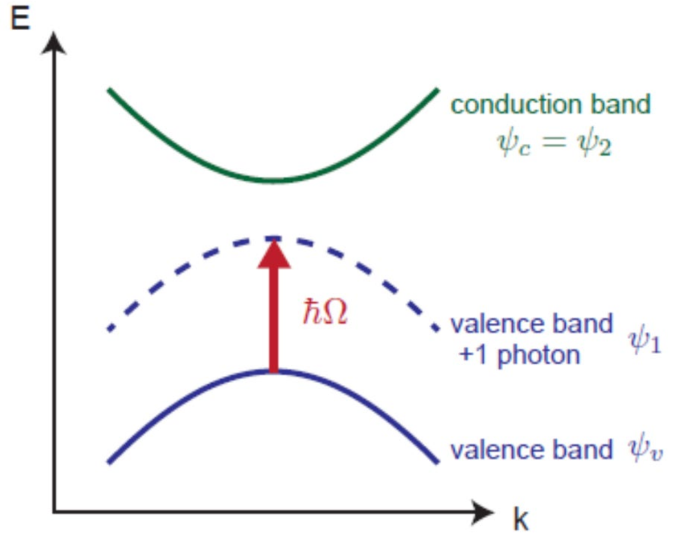
Non-radiative relaxation

Shift-current

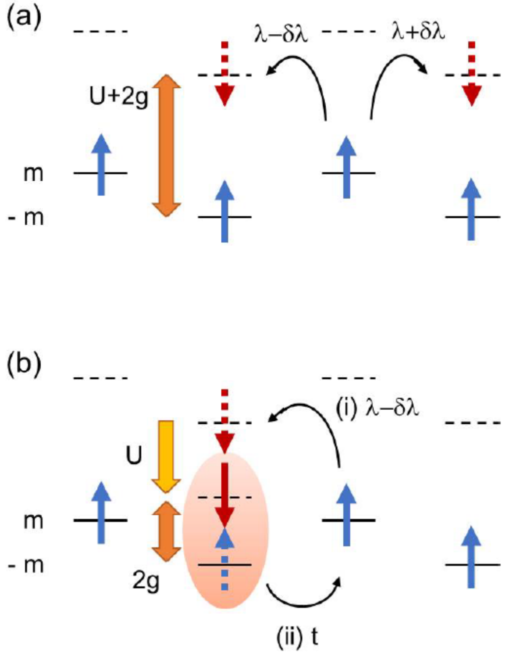
Even order response to E

Shift current of excitons and electromagnons

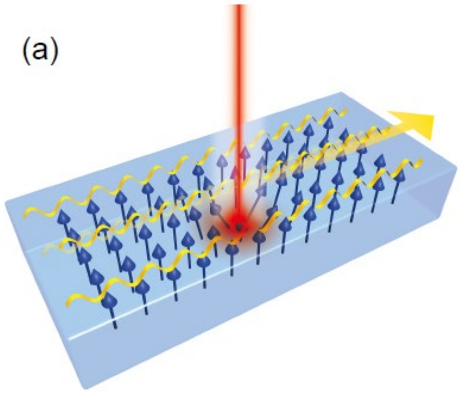
Electron correlation → exciton formation in band insulator and magnetism in Mott Insulator



T. Morimoto and N. Nagaosa,
Phys. Rev. B **94**, 035117(2016).



T. Morimoto and N. Nagaosa
Phys. Rev. B**100**, 235138 (2019)



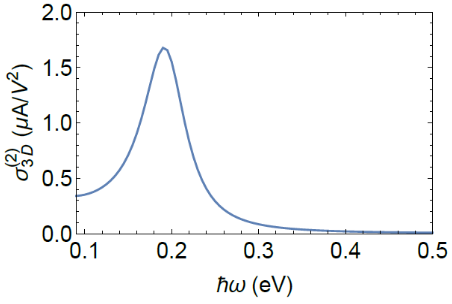


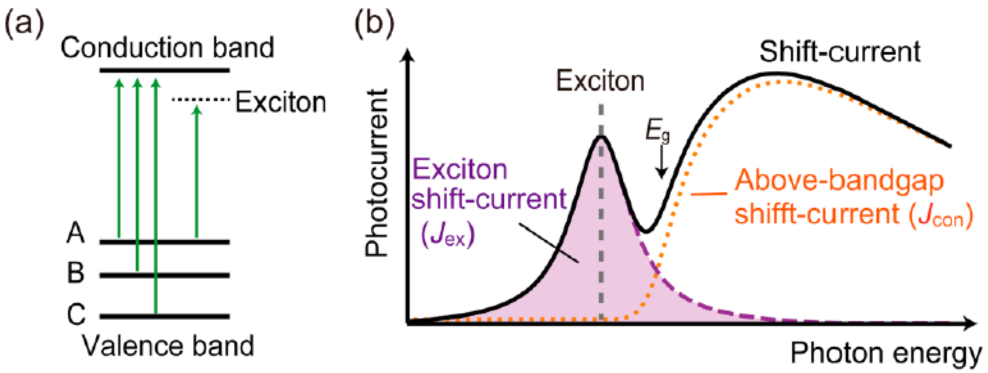
FIG. 4. Nonlinear conductivity $\sigma^{(2)}_{3D}(\omega)$. We used the parameters $t = 0.25$, $m = 0.1$, $U = 1.5$, $g = 0.25$, $\lambda = 0.05$, $\delta\lambda = 0.02$, $\gamma = 0.03$ (the units are eV). The peak around $\hbar\omega \simeq 0.2$eV corresponds to the electromagnon excitation at $q = 0$, and indicates that the electromagnon excitation supports nonvanishing dc current response.

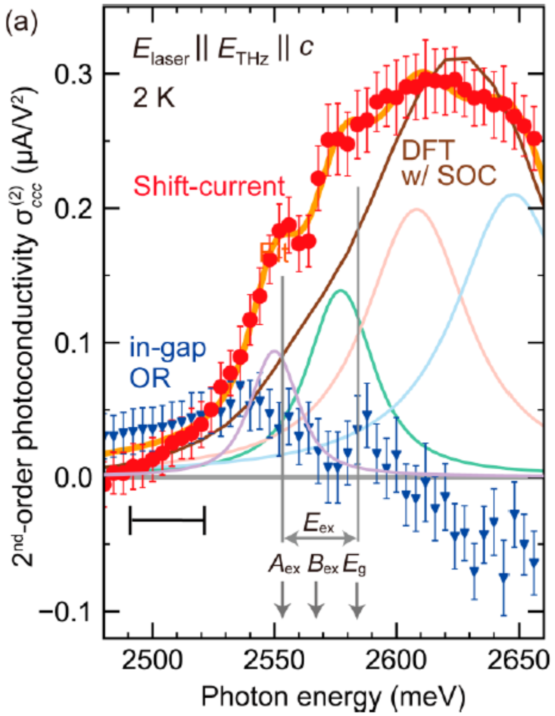
Exciton and electromagnon also support the dc shift current

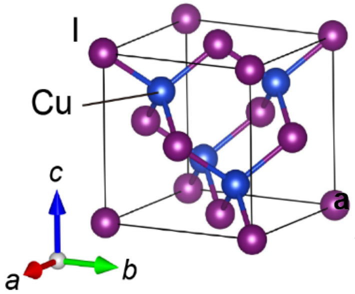
Shift current of excitons

CdS

Sotome et al. PRB2021

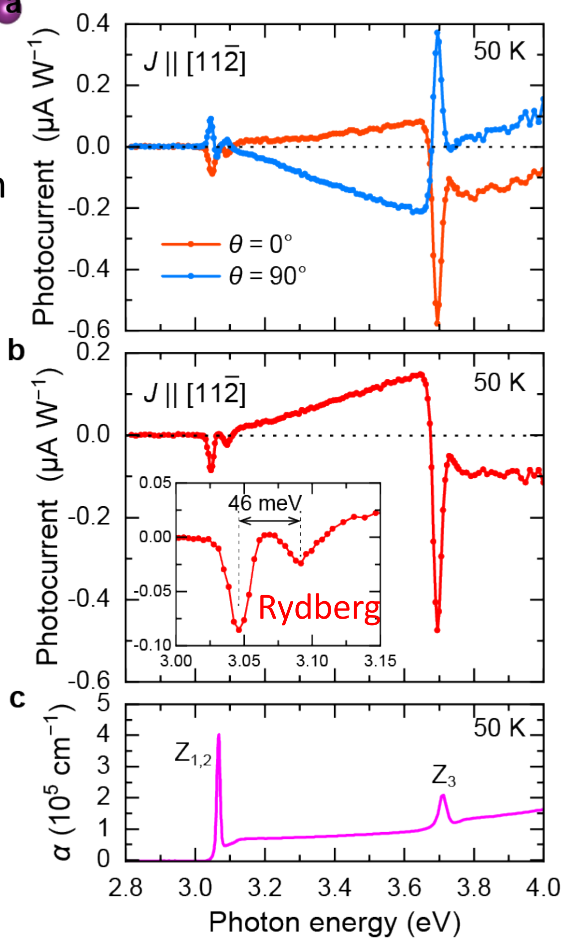






Zincblende with broken-inversion along (111)

$$J_{\|[11\bar{2}]} = \frac{\sqrt{6}}{3} I_0^2 \alpha \cos 2\theta$$

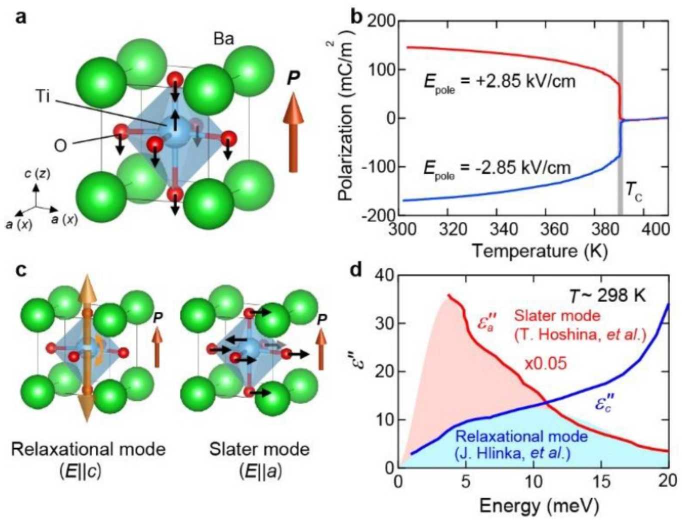


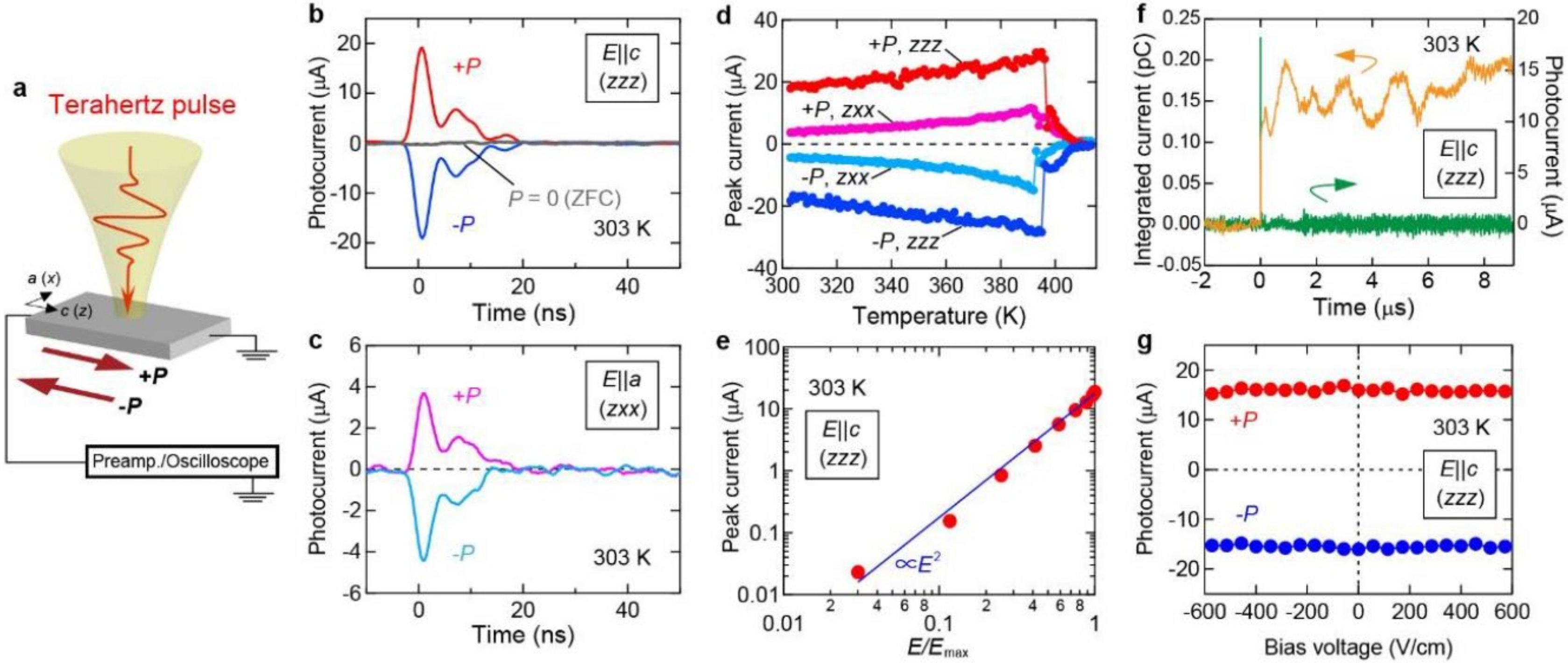
Opposite shift current for exciton vs. continuum

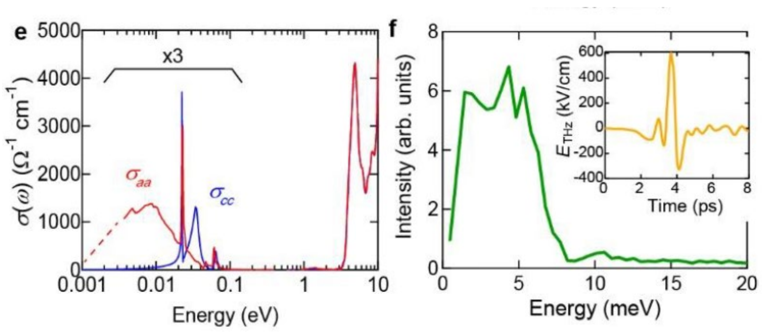
M. Nakamura, M. Kawasaki, G.Y. Guo *et al.*, Nature Communications 15 (1), 9672 (2024)

Shift current by phonon excitation in BaTiO3

Y. Okamura et al. arXiv:2201.00133/PNAS 2022







DC component of photo-current due to phonon excitation in ferroelectrics
Photo-current without photo-carriers in insulator

THz region
phonon excitation

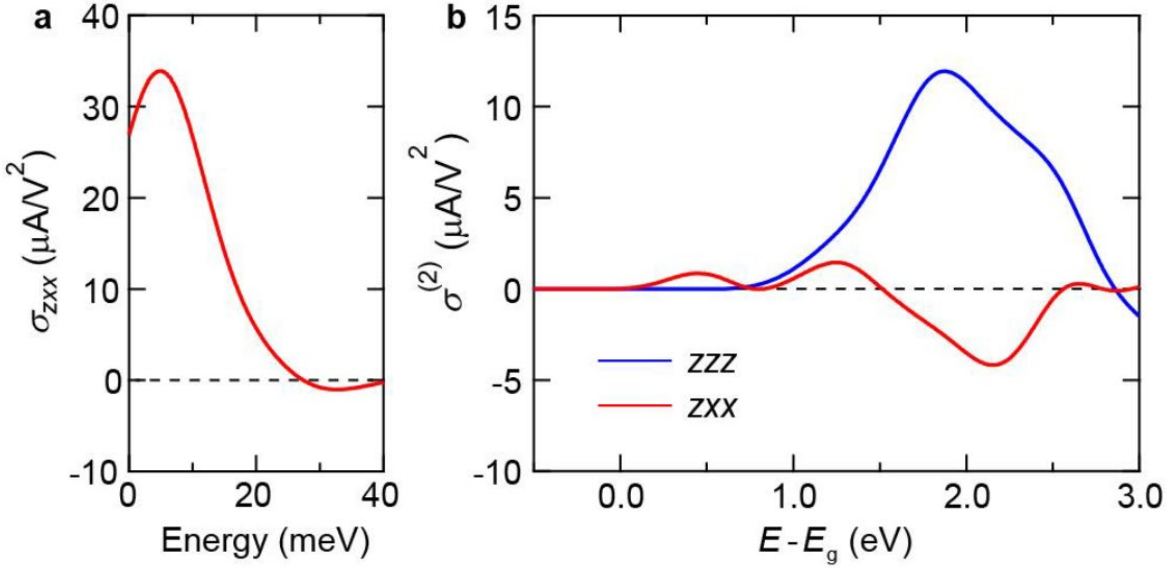
Energy gap
~ 3eV

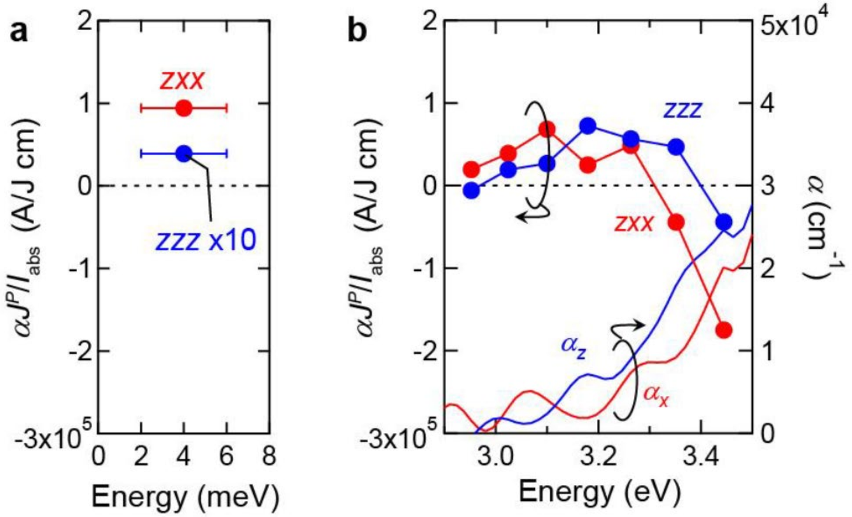
First-principles calculation of shift current in BaTiO3

$$\Pi_{abc}(\omega)=\frac{\pi\hbar^2}{V_c}\sum_v \mathrm{Im}\left[A_1^b(\omega)A_2^{ac}(\omega)-A_1^c(-\omega)A_2^{ab}(-\omega)\right]\delta(\omega-\omega_v),$$

$$A_1^b(\omega)=\sum_{\mathbf{k}}\frac{1}{N_kV_c}\sum_{mn}f_{mn}\frac{g_{mn}v_{nm}^b}{\hbar(\omega-\omega_{nm})},$$

$$A_2^{ac}(\omega)=\sum_{\mathbf{k}}\frac{1}{N_k}\left[\sum_{\substack{mn\\ l\neq m}}\frac{v_{ml}^a v_{ln}^c g_{mn}}{\hbar^2\omega_{lm})}\left(\frac{f_{mn}}{\omega_{nm}-\omega}-\frac{f_{ln}}{\omega_{nl}-\omega}\right)+\sum_{\substack{mn\\ l\neq m}}\frac{(v_{ml}^a v_{ln}^c g_{mn})^*}{\hbar^2\omega_{lm})}\left(\frac{f_{mn}}{\omega_{nm}+\omega}-\frac{f_{ln}}{\omega_{nl}+\omega}\right)\right]$$





ARTICLES
https://doi.org/10.1038/s41567-021-01465-z

nature physics

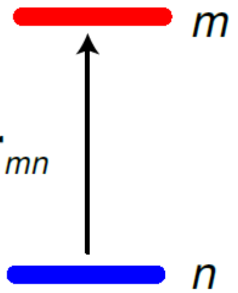
Riemannian geometry of resonant optical responses

Nature Physics **18**, 290 (2022)

c.f. Y. Gao, S.A. Yang, and Q. Niu, PRL2014
D. Kaplan, T. Holder, and B. Yan, 2022

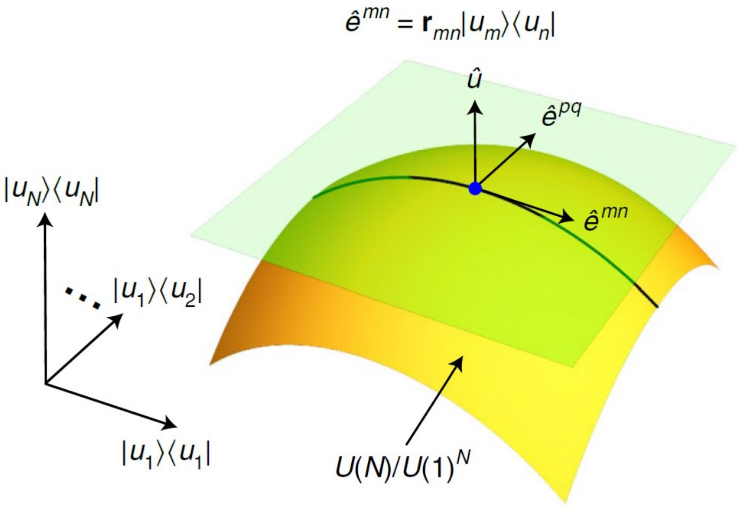
Junyeong Ahn[1], Guang-Yu Guo[2,3], Naoto Nagaosa[4,5] and Ashvin Vishwanath[1]

$$\langle \psi_m | \hat{H}_{\rm int} | \psi_n \rangle = e\mathbf{E}\cdot\mathbf{r}_{mn}$$



$$\begin{array}{l} \text{Metric} \\ \text{Connection} \\ \text{Curvature} \end{array} \begin{bmatrix} Q^{mn}_{cb} \\ C^{mn}_{cab} \\ K^{mn}_{cbad} \end{bmatrix} = r^c_{nm} r^b_{mn} \times \begin{bmatrix} 1 \\ (R^m_a - R^n_a)/i \\ (F^m_{ad} - F^n_{ad})/i \end{bmatrix} \begin{array}{l} \rightarrow \text{Transition rate } (dN/dt) \\ \rightarrow \text{Direct current generation } (dR/dt) \\ \rightarrow \text{Photovoltaic hall effect } (d\sigma_{\rm Hall}/dt) \end{array}$$

b Tangent basis vectors = transition dipole moment matrix elements



$$\sigma^{c;ab}_{\rm shift} = \frac{\pi e^3}{2\hbar^2} \sum_{m,n} \int_{\mathbf{k}} \delta(\omega - \omega_{mn}) f_{nm} i (C^{mn}_{bca} - (C^{mn}_{acb})^*)$$

second order shift current

$$\sigma^{d;abc}_{\rm inj} = \frac{\pi e^4}{6\Gamma\hbar^3} \sum_{m,n} \int_{\mathbf{k}} \delta(\omega - \omega_{mn}) f_{nm} i K^{mn}_{cbad} + \ldots,$$

third order optical conductivity

Geometry of transitions between Bloch states in solids

$$r^a_{mn}(\mathbf{k}) = \delta_{mn} i\partial_a + \langle u_{m\mathbf{k}}|i\partial_a|u_{n\mathbf{k}}\rangle$$ matrix Berry phase

$$\hat{e}^{mn}_a(\mathbf{k}) \equiv r^a_{mn}(\mathbf{k})|u_{m\mathbf{k}}\rangle\langle u_{n\mathbf{k}}|$$ base of tangential space

$$Q^{mn}_{ba} \equiv (\hat{e}^{mn}_b, \hat{e}^{mn}_a) = r^b_{nm} r^a_{mn}$$ inner product and quantum metric

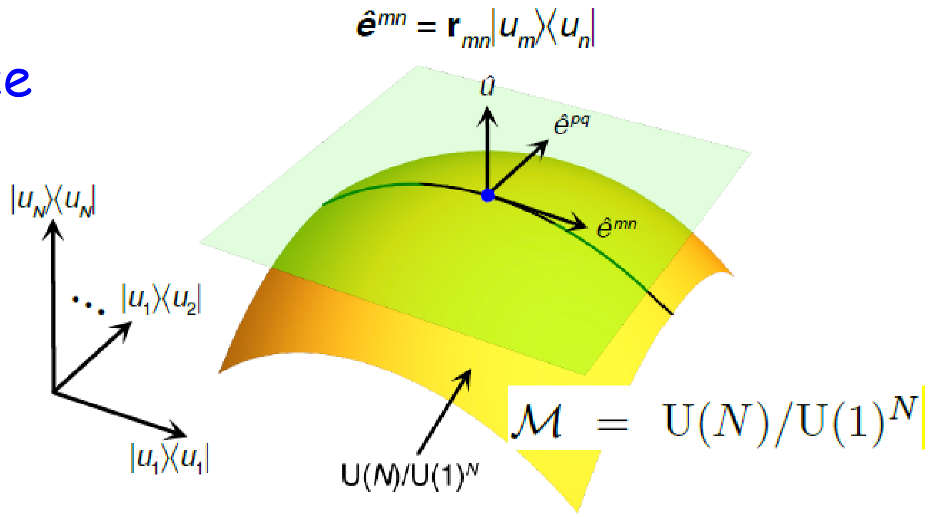
$$\partial_c \hat{e}^{mn}_a = \sum_b C^{mn|b}_{ca} \hat{e}^{mn}_b + \dots$$

$$C^{mn}_{bca} \equiv \sum_e Q^{mn}_{be} C^{mn|e}_{ca} = (\hat{e}^{mn}_b, \nabla_c \hat{e}^{mn}_a) = r^b_{nm} r^a_{mn,c}$$ connection

$$D^{mn}_{badc} \equiv (\hat{e}^{mn}_b, \nabla_d \nabla_c \hat{e}^{mn}_a) = r^b_{nm} r^a_{mn,cd}$$ curvature

$$K^{mn}_{badc} = D^{mn}_{badc} - D^{mn}_{bacd} = -i r^b_{nm} [\mathcal{F}_{dc}, r^a]_{mn}$$

Tangent basis vectors = Transition dipole moment matrix elements

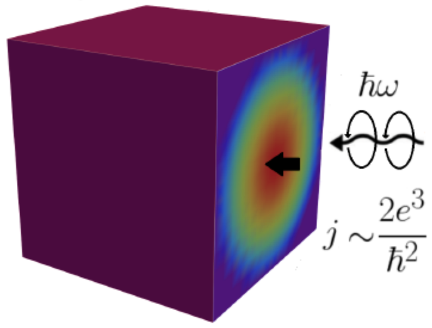


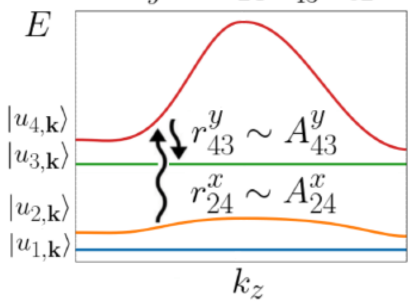
$$(A, B) = \sum_{\alpha,\beta} A^*_{\alpha\beta} B_{\alpha\beta} = \mathrm{Tr}\left[A^\dagger B\right]$$

Quantized Integrated Shift Effect in Multigap Topological Phases

Wojciech J. Jankowski[1,*] and Robert-Jan Slager[1,†]

Phys. Rev. Lett. 133, 186601 (2024)





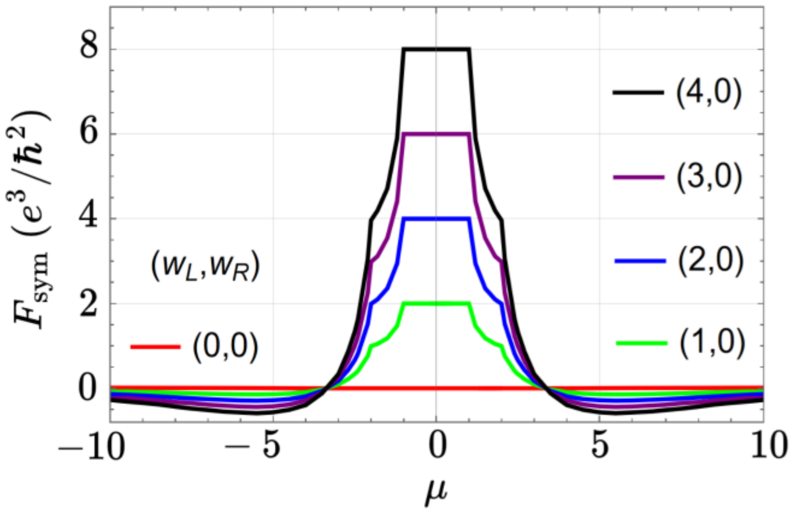
$$\sigma^{cab}_{\text{shift}}(\omega) = \frac{\pi e^3}{2\hbar^2}\sum_{m,n}\int_{\text{BZ}}\frac{\mathrm{d}^3\mathbf{k}}{(2\pi)^3}\delta(\omega-\omega_{mn})f_{nm}\mathrm{i}\left(C^{mn}_{acb}-C^{nm}_{bca}\right).$$

$$T^{mn}_{abc} \equiv C^{mn}_{abc} - C^{mn}_{acb}.$$

$$\int_0^{\omega_{\max}} \mathrm{d}\omega\ [\sigma^{abc}_{\text{shift,C}}(\omega)+\sigma^{bca}_{\text{shift,C}}(\omega)+\sigma^{cab}_{\text{shift,C}}(\omega)]$$
$$= -\frac{e^3}{8\pi^2\hbar^2}\int_{\text{BZ}}\mathrm{d}^3\mathbf{k}\sum_{m,n} f_{nm}(T^{mn}_{abc}+T^{mn}_{bca}+T^{mn}_{cab}),$$

$$F_{\text{sym}} \equiv -\mathrm{i}\int_0^{\omega_{\max}} \mathrm{d}\omega\ [\sigma^{xyz}_{\text{shift}}(\omega)+\sigma^{yzx}_{\text{shift}}(\omega)+\sigma^{zxy}_{\text{shift}}(\omega)]$$
$$= \frac{e^3}{4\pi^2\hbar^2}\int_{\text{BZ}}\mathrm{d}^3\mathbf{k}\ (\mathbf{a}^v\cdot\mathbf{E}\mathbf{u}^v+\mathbf{a}^c\cdot\mathbf{E}\mathbf{u}^c),$$

For real Bloch w.f. Fsym is related to non-Abelian top. Number



PNAS RESEARCH ARTICLE | PHYSICS OPEN ACCESS

Photon-drag photovoltaic effects and quantum geometric nature

Ying-Ming Xie[a,1] and Naoto Nagaosa[a,b,1]

PNAS 122, e2424294122 (2025)

Finite wavevector of photon
Photovoltaic effect in centrosymmetric materials

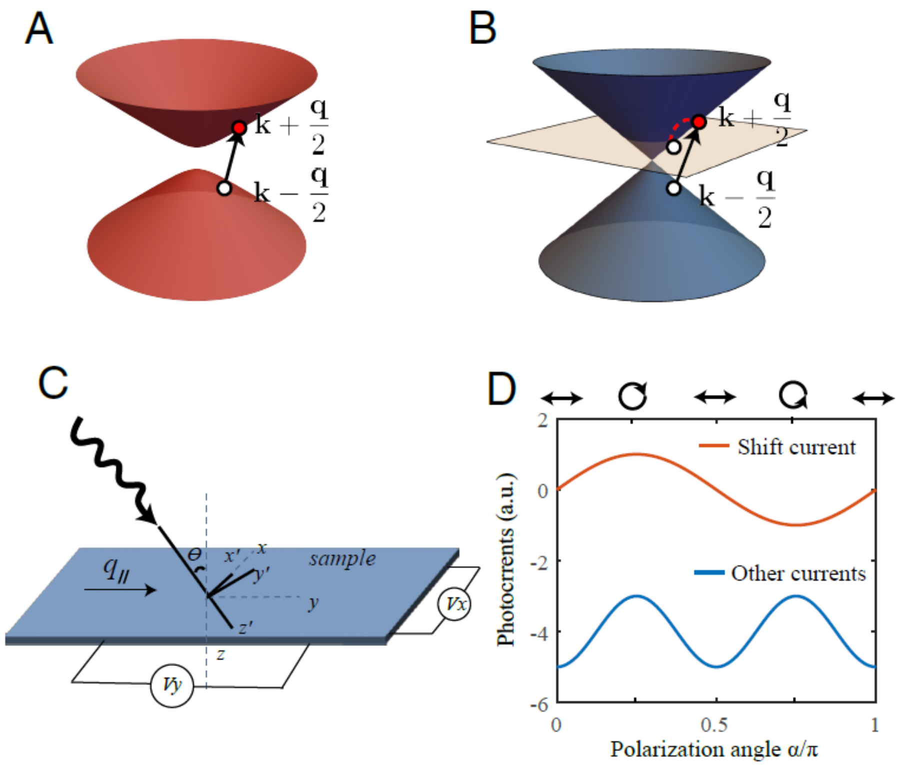


Fig. 1. (*A* and *B*) schematically show the nonvertical photoexcitations in insulators (or semiconductors) and semimetals due to the presence of finite photon-momentum $\boldsymbol{q}$. In the presence of Fermi surfaces, the intraband photoexcitations appear [red dashed line in (*B*)]. (*C*) A light incident onto a sample with an angle θ, where xyz ($x'y'z'$) labels the sample (incident light) frame. (*D*) schematically show that the photon-drag induced shift current follows a polarization dependence of $C\sin(2\alpha)$, while other currents show $L_1\sin(4\alpha) + L_2\cos(4\alpha) + D$ in nonmagnetic centrosymmetric materials.

Table 1. Quantum geometric nature of photon-drag BPVE

Photon-drag response	L-shift	C-shift	L-inj/fs	C-inj/fs
Geometry quantity	$\mathcal{G}$, $\partial_k\Omega$	Ω, $\partial_k\mathcal{G}$	$\mathcal{G}$, $\partial_k\mathcal{G}$	Ω, $\partial_k\Omega$
T condition	X	—	—	X

Here, $\mathcal{G}$ and Ω represent the quantum metric tensor and the Berry curvature, respectively. The cross symbol indicates that time-reversal symmetry must be broken.

Summary and Perspectives

1. Geometric view of quantum physics - ubiquitous due to wave-particle duality
2. Berry phase as an emergent electromagnetic field
3. Quantum metric as an emergent gravity
4. Electrons in solids as an ideal laboratory to study emergent field theory
5. Linear and nonlinear responses as manifestations of quantum geometry
6. Topological states as a global aspect of quantum geometry
7. Non-Abelian generalization
8. Higher-dimensional spaces including r,k, c1,c2,,,,,,